Take-Home Test 2

Due Thursday, December 6

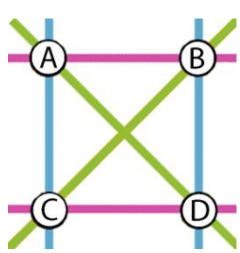
Instructions: The following will be graded as a substitute for an in-class Test 2; but the instructions are the same as for regular semester homework. You may consult other sources, or discuss with other students in the class; nevertheless the work that you submit must be your own, not copied.

1. Let A and B be the points $(-1,0)$ and $(1,0)$ in the Euclidean plane. Denote by AP and BP the distances from P to the points A and B respectively. Fix a positive real number λ, and let γ be the curve consisting of all points P such that $\frac{BP}{AP} = \lambda$.
   a. Find the equation of the curve γ in Cartesian coordinates (x, y).
   b. In words, give a simple geometric description of γ.
   c. Explain the relevance of this particular choice of curve γ in the context of our course, describing the relation of γ to the particular choice of points A and B.

In any inversive plane, if we fix a point N (which we may think of as the north pole, although the choice of N is arbitrary), it is not hard to see that the points other than N, together with the circles through N (after removing N) always form an affine plane. This fact follows directly from the axioms. In the classical case of the real inversive plane, this observation can also be realized using stereographic projection through N as described in class. Given an affine plane, we can sometimes (although not always) reverse the process just described, in order to construct an inversive plane from an affine plane by the addition of one extra point (which we might call ∞) and with a judicious choice of circles. We have seen how the real inversive plane can be found to arise in just such a manner. The finite classical inversive planes provide further examples. Before proceeding with examples, we derive formulas for counting points and circles in finite inversive planes.

Given a finite inversive plane **IP**, and fixing an arbitrary choice of one of its points N, the remaining points and the circles through N (after removing N) form a finite affine plane **AP** of order n, say. This means that **IP** has exactly n^2+1 points (including N); also N lies on n^2+n circles of **IP**; and every circle through N has exactly $n+1$ points (including N). Since these numbers are the same regardless of the point N chosen, it must in fact be the case that **IP** has n^2+1 points; every circle has $n+1$ points; every point lies on n^2+n circles; and every pair of distinct points has exactly $n+1$ points in common. The total number of circles in **IP** must be $n(n^2+1)$. (For if m is the total number of circles, there are $(n+1)m = (n^2+1)(n^2+n)$ incident pairs (P,γ) where γ is a circle and P is a point on γ). The two smallest cases $n=2$ and $n=3$ are described below.

We may extend the affine plane of order 2 to form an inversive plane with exactly 5 points, in a fashion very reminiscent of the way a one-point extension of the Euclidean plane yields the real inversive plane. Starting with the affine plane

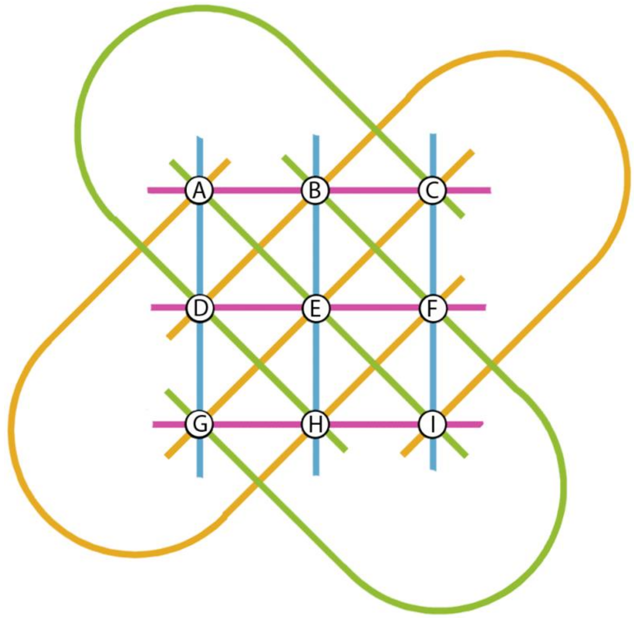


we add a new point called ∞ to every line; we also add four affine 'circles' to obtain altogether five points A, B, C, D, ∞ and ten circles

$$ABC, ABD, ACD, BCD, AB\infty, AC\infty, AD\infty, BC\infty, BD\infty, CD\infty.$$

This is the smallest inversive plane, mentioned in class. For your benefit, you should convince yourself that this does indeed satisfy the three axioms of inversive plane geometry. There is nothing very deep here because it consists simply of all $\binom{5}{3} = 10$ triples on 5 points.

Rather less trivial is the next-smallest inversive plane, constructed by extending the affine plane of order 3, by the addition of a new point called ∞, as outlined in Question 2 below.



2. Begin with the affine plane of order 3 as shown on the right. The original nine points $A, B, \ldots, I$ are called the *affine points*.

   Complete the following blanks appropriately in order to extend the affine plane of order 3 to an inversive plane of order 3, using the axioms and the counting formulas above as necessary:

   There are two types of circles: extended affine lines (consisting of affine lines, with the new point ∞ added) and affine circles (consisting of affine points only, no ∞). The total number of circles should be ____________________________. Each circle contains _______________ points, and every point lies in exactly _________________ circles. Every pair of distinct points lies in exactly ____________________ circles.

   The number of circles formed by extending affine lines (by the addition of the new point ∞) is _________________. So the number of affine circles is _______________________ . Any two of these circles meets in at most _______________ points.

   The number of ways to choose four affine points with no three collinear, is ________________ . (So these can't all be circles of our inversive plane; we will have to choose a subset of these as our affine circles.)

   Assuming that $ABDE$ is one of our affine circles, the others must be:

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3. Miquel's Theorem plays a role (in inversive plane geometry) analogous to the role of the theorems of Pappus and Desargues in affine and projective plane geometry. In an inversive plane, we say that a quadruple of points $ABCD$ is concyclic if the four points A, B, C, D all lie on the same circle. Now suppose we have eight distinct points A, B, C, D, E, F, G, H in an inversive plane; and consider the six quadruples of points $ABCD,\ ABEF,\ BCFG,\ CDGH,\ ADEH,\ EFGH.$ Miquel's Theorem says that in the classical case (i.e. in an inversive plane coordinatized by a field), if at least five of these quadruples are concyclic, then so is the sixth quadruple. The following converse holds: If Miquel's condition holds for every set of eight distinct points in an inversive plane, then the plane must be classical (i.e. the plane must be coordinatized by a field).

   Miquel's configuration may be visualized using our model of the real inversive plane based on a Euclidean sphere S. Taking the eight points A, B, C, D, E, F, G, H to be the vertices of a cube inscribed in S, the six concyclic quadruples of vertices are those of the six faces of the inscribed cube. By stereographic projection, this gives rise to a Miquel configuration in the 'flat' model of the real inversive plane with point set $\mathbb{R}^2 \cup \{\infty\}$.

   a. Using ample space on a full sheet of blank paper, provide a straightedge and compass construction demonstrating the validity of Miquel's Theorem for a particular choice of eight distinct points A, B, C, D, E, F, G, H, all on your paper. Label these eight points of the configuration. (Of course this is not a proof of Miquel's Theorem, nor are you asked to provide such a proof, only a construction illustrating the theorem in a particular case.)

   b. Using another sheet of paper, provide a straightedge and compass construction of another Miquel configuration such that A is the point at infinity; and the seven points B, C, D, E, F, G, H all lie on your page.