Solutions to Take-Home Test 2

1. (a) The point $P(x,y)$ lies on the curve γ iff

$$0 = BP^2 - \lambda^2 AP^2 = (x-1)^2 + y^2 - \lambda^2[(x+1)^2 + y^2].$$

If $\lambda = 1$, this simplifies to $x = 0$ (the y-axis); whereas if $\lambda \neq 1$, we complete the squares to rewrite this as

$$(x-a)^2 + y^2 = r^2$$

where $r = \frac{2\lambda}{|1-\lambda^2|}$ and $a = \frac{1+\lambda^2}{1-\lambda^2}$.

(b) If $\lambda \neq 1$, γ is the Euclidean circle centered at $C(a,0)$ with radius r (where a and r are given above). If $\lambda = 1$, then γ is a line (the y-axis).

(c) In either case, γ is a circle in the real inversive plane, inverting $A \leftrightarrow B$. As λ varies over all positive real numbers, γ varies over all such circles. For $\lambda \neq 1$, we note that

$$AC \cdot BC = |a+1| \cdot |a-1| = |a^2 - 1| = r^2$$

which shows that γ does indeed invert $A \leftrightarrow B$. For $\lambda = 1$, γ is the y-axis, which inverts (i.e. reflects) $A \leftrightarrow B$. In the limit as $\lambda \to 1$, note that γ has radius $r \to \infty$; also its center $C \to \infty$.

2. There are two types of circles: extended affine lines (consisting of affine lines, with the new point ∞ added) and affine circles (consisting of affine points only, no ∞). The total number of circles should be $n(n^2+1) = 30$. Each circle contains $n+1 = 4$ points, and every point lies in exactly $n(n+1) = 12$ circles. Every pair of points lies in exactly $n+1 = 4$ circles.

The number of circles formed by extending affine lines (by the addition of the new point ∞) is 12. So the number of affine circles is $30 - 12 = 18$. Any two of these circles meets in at most 2 points.

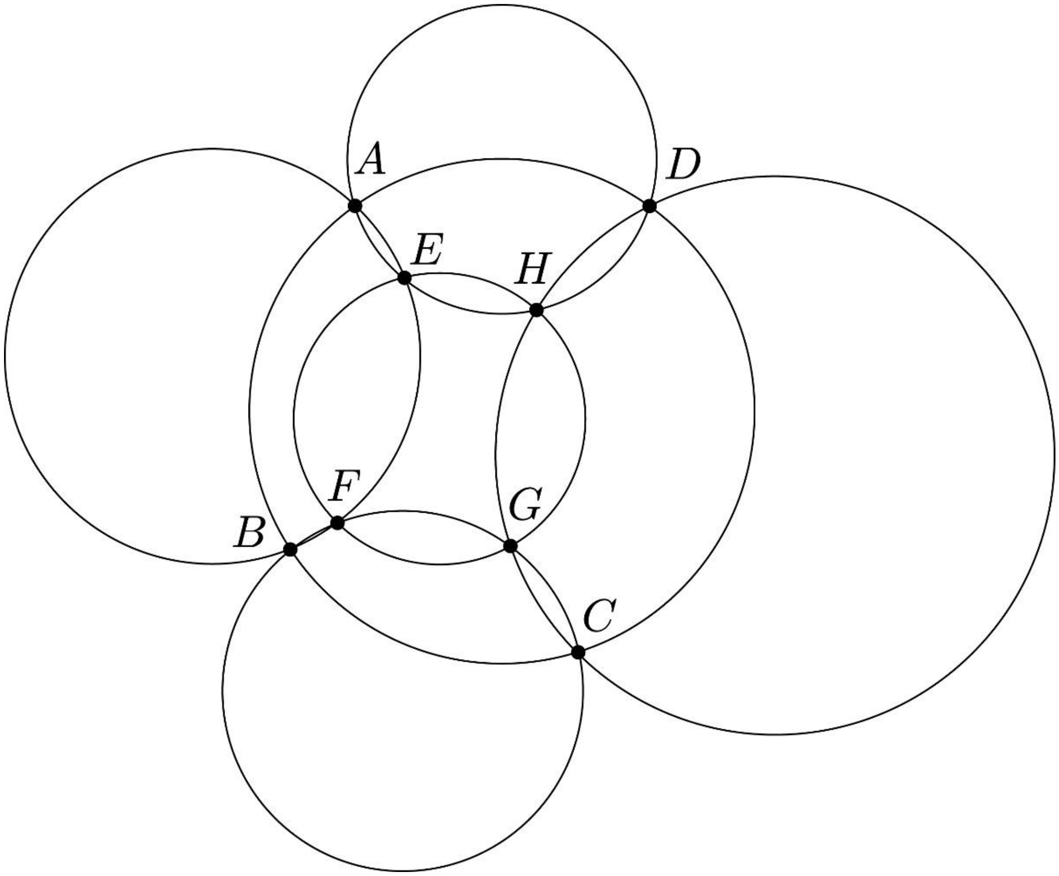
The number of ways to choose four affine points with no three collinear, is $(9*8*6*3)/(4*3*2*1) = 54$. (So these can't all be circles of our inversive plane; we will have to choose a subset of these as our affine circles.)

Assuming that $ABDE$ is one of our affine circles, the others must be:

$ABGH$	$ABFI$	$ACDF$	$ACEH$	$ACGI$	$ADHI$
$AEFG$	$BCDG$	$BCEF$	$BCHI$	$BDFH$	$BEGI$
$CDEI$	$CFGH$	$DEGH$	$DFGI$	$EFHI$	

3. Typical examples of configurations in (a) and (b) appear on the following page.

3(a)



3(b)