
# Solutions to Test 1

## Section A: True/False *(3 points each)*

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|---|---|---|---|---|---|---|---|---|---|
| 1. F | 2. F | 3. F | 4. T | 5. T | 6. T | 7. F | 8. T | 9. F | 10. F |

*In Section A you were not expected to provide explanation; however, here are some comments for your benefit.*

1. There are many currently unsolved problems in Euclidean plane geometry, including some that are known to be algorithmically unsolvable.
2. To obtain an affine plane, you should remove a line, not a quadrilateral.
3. As described in class, the original works of Euclid, while ahead of the time, were imprecise and clunky. Euclid also failed to clearly distinguish the role of undefined notions. The work of Hilbert and others in the early 20th century cleaned up the axiomatic treatment of plane geometry, resulting in something we find readable today. For example the famous Postulate V of Euclid may be translated as '*That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles*.' In the modern treatment, we replace this with Playfair's Axiom (our axiom A2), a simpler statement which conveys the same critical information.
4. This is easy to prove; and it *should* be true, since it is the dual of the third axiom (P3). (In order for the principle of duality to hold, the dual of every true statement should also be true.)
5. If $n \geq 3$ then three collinear points lie on more than one plane; whereas if $n = 2$ or three given points are not collinear, they lie in a unique plane. Recall our discussion and demonstration of this fact for $F = \mathbb{F}_3$ and $n = 2,3,4$ using the deck of Set® cards: there we took three cards not forming a 'set' (i.e. not collinear, and thus rather forming a triangle) and found the unique plane containing that triple of cards.
6. This is the usual construction of the real projective plane.
7. In affine plane geometry, 'point' and 'line' are undefined terms; and 'shortest' is meaningless without a notion of distance.
8. In an affine plane, every point is on $n + 1$ lines where the order $n \geq 2$ (and $n$ may be infinite).
9. In the axioms for plane geometry, 'point' and 'line' are undefined terms; and algebraic curves cannot be defined unless the plane is classical (i.e. coordinatized by a field).
10. In the card game SpotIt®, cards and symbols represent points and lines of the classical projective plane over $\mathbb{F}_7$.

## Section B

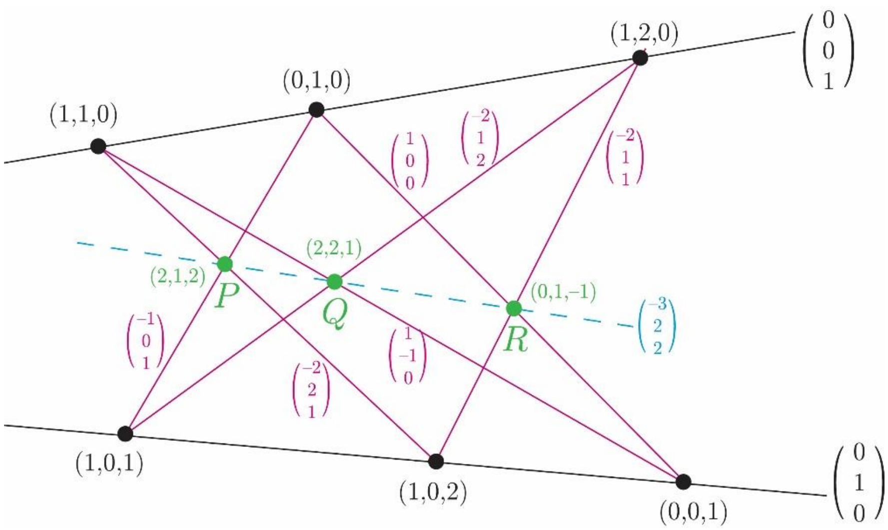
11. *(32 points)* The classical theorem of **Pappus** holds in the Euclidean plane only if one is careful to state the exceptional cases that arise due to the possibility of **parallel** lines. If one adds a point at **infinity** to every **parallel** class of lines in the Euclidean plane, then joins all the new points with

a single new **line** , one obtains a classical **projective** plane. In this new plane, the classical theorem of **Pappus** holds without exception because no two lines are **parallel** .

The Euclidean plane is an example of a classical **affine** plane; but because it is coordinatized by the real numbers, it admits several features not found in a more general **affine** plane. Examples of these properties include **distance** between two points; the **order** of points on a line; and the **angle** between two intersecting lines.

In the Euclidean plane, a **conic** has the property that no three of its points are **collinear**. This property can be proved, not directly from the axioms, but rather using **algebra**.

12.




13. Suppose a plane has the following property: Given any four lines $a, b, c, d$, no three of which are concurrent, let $P = a \cap b$, $Q = a \cap c$, $R = a \cap d$, $S = b \cap c$, $T = b \cap d$; $U = c \cap d$; then the three lines $PU, QT, RS$ are concurrent. Then the plane is classical.

## Section C

14. Since $2S$ (the figure $S$ scaled by a factor of 2 in all directions) splits naturally into three copies of $S$, the figure $S$ has dimension $d$ satisfying $3 = 2^d$, i.e. $d = \frac{\ln 3}{\ln 2} \approx 1.58496$. Note that $1 < d < 2$ as expected.