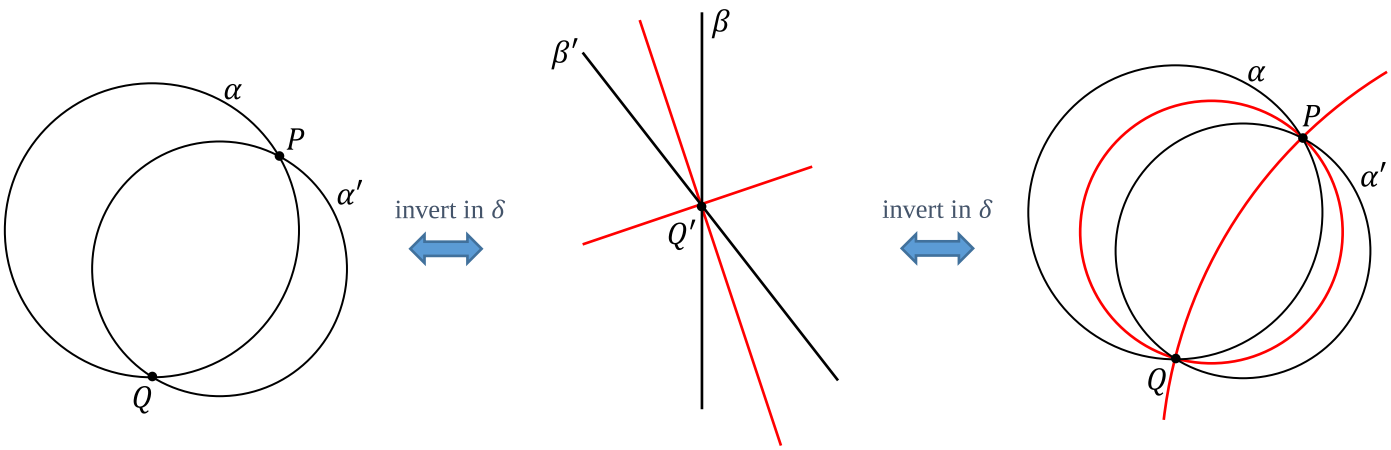
Solutions to SAMPLE Final Exam

1. (*10 points*) Let P and Q be the two points of intersection of α and α'. Perform an inversion in a circle δ centered at P. This maps α and α' to circles β and β' passing through $P' = \infty$ and Q' as shown. Thus β and β' are ordinary lines passing through Q', the inverse of Q in δ. Now there are two lines (shown here in red) through Q' which reflect β to β'; these two red lines are the bisectors of the angles at Q' formed by β and β'. Invert again in δ to obtain two possible choices (shown again in red) for a circle inverting α to α'.



2. a. This curve has degree 3.

   b. By Bezout's Theorem, the curve has at most 3 collinear points. (Alternatively, to find intersection points of the curve, requires simultaneously solving two polynomial equations, one of degree 3 and one of degree 1. Use the second equation to write $y = mx + b$; then substitute into the first equation to obtain a polynomial equation for x of degree 3. This has at most 3 real roots, so there are at most three points of intersection.)

3. The unique conic in the real projective plane passing through the five points $(1,0,0), (0,1,0), (0,0,1), (1,-1,1), (1,2,4)$ has equation $2xy + xz - yz = 0$. To obtain this equation, first consider an arbitrary conic $ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$ where a, b, c, d, e, f are real constants, not all zero. Assuming that the five points given all lie on this conic, we obtain five linear equations in the six unknowns; and solving this system gives the equation $2xy + xz - yz = 0$ up to a scalar multiple.

4. (a) In the Euclidean plane, given a point P not on a line ℓ, there is a unique line through P not meeting ℓ. In the hyperbolic plane, there are instead infinitely many lines through such a point P not meeting the line ℓ.

   Also in the Euclidean plane, every triangle has angle sum equal to $\pi = 180°$; in the hyperbolic plane, every triangle has angle sum *less than* $\pi = 180°$.

Furthermore, in the Euclidean plane, a circular disk of radius r has area πr^2; in the hyperbolic plane, such a disk has area greater than πr^2.

(b) Unlike the Euclidean plane, large triangles in physical space have angle sum different from $180°$; but the actual angle sum in any particular triangle depends not only on the side lengths of the triangle—it also depends on the position of the triangle relative to nearby masses, which affect the angle sum.

In fact, distances between points in physical space cannot be absolutely measured without agreeing on synchronicity of events. This again differs from distances in the Euclidean plane, which are well-defined and time-independent.

There are also differences which only become evident on the very small scale, smaller than our usual practical range of measurement. In the Euclidean plane, given any three distinct collinear points, one of the points is between the other two. This well-defined relation of order of points on a line, fails for physical space; and the failure becomes evident at very small distances—in fact at very small distance scales, distance no longer has physical meaning. This statement does not refer to the lack of precision intrinsic in our tools of measurement; rather, it is a feature of space (or rather spacetime) itself.

(c) The hyperbolic plane is uniform (i.e. homogeneous) in a way that physical space is not. For example, in the hyperbolic plane, any two triangles with given side lengths a, b, c must have the same angles. As described in (b), this is not true for physical space, where the angle defect of a triangle depends on the distribution of nearby mass.

5. A projective plane has points, lines and incidence satisfying three axioms:
   (P1) Any two distinct points lie on a unique line.
   (P2) Any two distinct lines meet in a unique point.
   (P3) There exists a set of four points, no three of which are collinear.

   Every projective plane satisfies also the dual of the third axiom, namely: There exists a set of four lines, no three of which are concurrent. To prove this, let A, B, C, D be a set of four points, no three of which are collinear; so the six lines $a = AB, a' = CD, b = BC, b' = AD, c = AC, c' = BD$ are distinct. Now it is easy to check that no three of the lines a, a', b, b' are concurrent. (For example, the point $a \cap b = \{B\}$ does not lie on $a' = CD$, otherwise the points B, C, D would be collinear. The other cases are similar.)

6. Given an affine plane (with affine points and affine lines) we obtain a projective plane as follows: We first add new points, one for each parallel class of affine lines: all affine lines in each parallel class are extended by the addition of the corresponding new point. Next, we join all these new points (which we call ‘points at infinity’) by adding a new line (called ‘the line at infinity’) which passes through all the points at infinity.

   Projective plane geometry is more natural in several respects: The axioms of projective plane geometry are simpler than those of affine plane geometry. For this reason, elementary theorems of projective plane geometry are typically easier to prove than the corresponding theorems of affine plane geometry, where care must be taken to accommodate special cases in which lines fail to intersect. Typically one theorem in projective geometry, translates into a large number of theorems in affine plane geometry, due to the large number of cases arising in the affine case due to parallel lines. Moreover, the principle of duality (valid in the projective plane, but not in the affine plane) means that many theorems in affine plane geometry exist in two forms, one the dual of the other; and in the projective setting, one simpler theorem suffices to cover both cases. Bezout’s Theorem (counting points of intersection of algebraic curves of degree m and n) provides a particularly elegant context in which the projective description (mn points of intersection) is simpler: the weakness of affine version (*at most* mn points of intersection) shows that without the 'points at infinity', the picture is incomplete.

7. Finite affine planes arise in the design of statistical experiments. We gave a hypothetical example of such an experiment, based on the affine plane of order 3.

   The finite projective plane of order 2 is used in the construction of the octonions, an 8-dimensional number system used in theoretical physics.

   Other applications have been mentioned briefly: the use of finite geometries in the construction of dense sphere-packings in $\mathbb{R}^n$ (which leads to the construction of good error-correcting codes); and elliptic curves in finite projective planes (cubic curves with an addition law as in HW3) are used in public key cryptography, primality testing and integer factorization; and in pseudorandom number generators.

8.

a. F	b. T	c. F	d. F	e. F	f. F	g. T	h. F	i. F	j. T

Comments in #8:

a. A pair of great circles on a sphere is an example of geodesics meeting at two points.

b. The angle sum of a 30°-45°-90° triangle is $165° < 180°$, so such a triangle exists in the hyperbolic plane. Moreover such a triangle tiles the plane by taking 12, 8 or 4 triangles meeting at each vertex.

c. In a dual inversive plane, any three distinct circles would have a unique point in common. This cannot occur in an inversive plane, where two circles can already be disjoint.

d. The real affine plane $\mathbb{R}^2$ satisfies the axioms for Euclidean plane geometry.

e. Each inversion reverses orientation in the inversive plane. So a composite of two inversions yields a transformation that preserves orientation; and this cannot itself be an inversion.

f. Any line in the real projective plane is topologically equivalent (i.e. homeomorphic) to a circle. Given three distinct points A, B, C on a circle, none of the three points is distinguished from the other two as lying 'in between'.

g. In the hyperbolic plane, an n-gon has area greater than 100 if n is large enough. (We would need $n \geq 34$ since an n-gon is decomposable into $n - 2$ triangles with angle sum less than $(n - 2)\pi$.)

h. This is a famous unsolved problem.

i. Stereographic projection distorts distances but preserves angles.

j. The real inversive plane can be modeled using the points of an ordinary sphere (i.e. the unit sphere in $\mathbb{R}^3$), which is orientable.