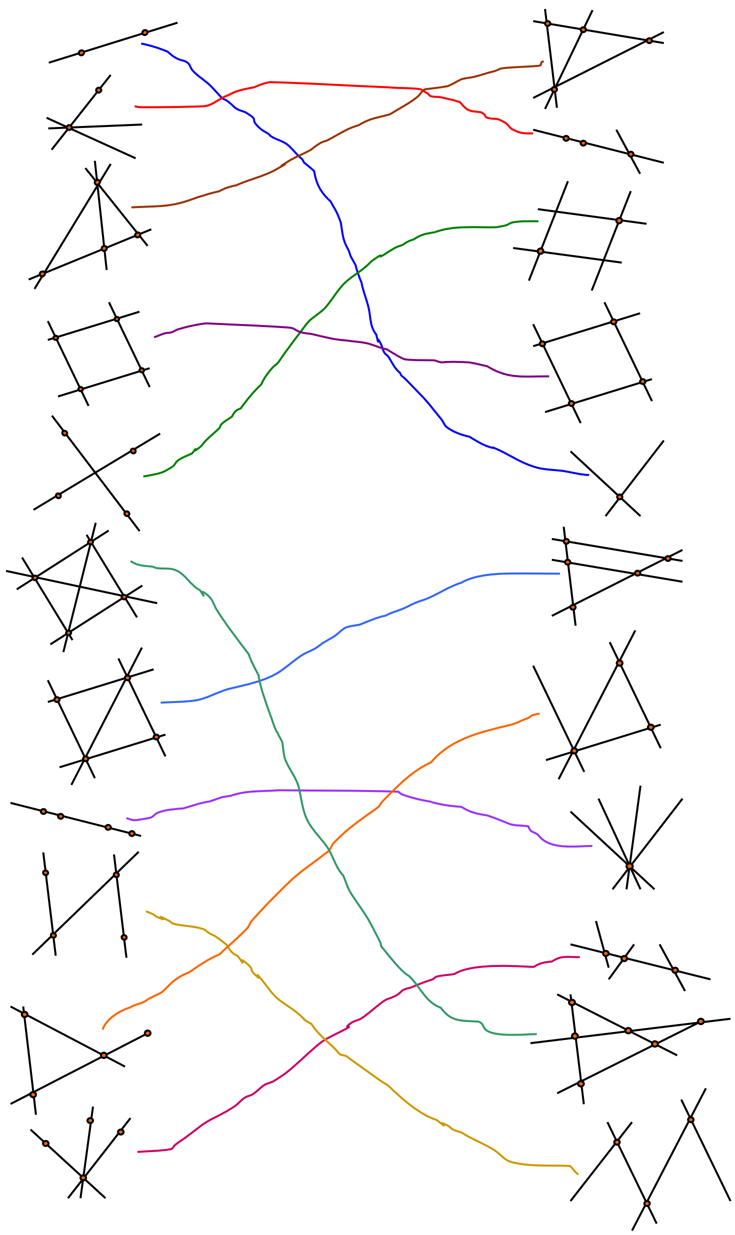
Solutions to the SAMPLE Test

Section A: True/False

1. F	2. T	3. T	4. F	5. T	6. F	7. T	8. F	9. F	10. F

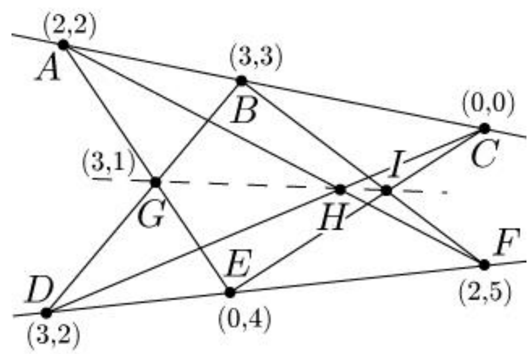
Section B

11. *(20 points)* Match each plane object on the left with its point-line dual on the right. The first one is done for you.



12. Scaling by a factor of $c = 0.1$ gives cA, the set of all real numbers in $[0,0.2]$ whose decimal digits are even. And the original set A is a disjoint union of five shifted copies of this: cA, $cA + 0.2$, $cA + 0.4$, $cA + 0.6$ and $cA + 0.8$. So A has Hausdorff dimension d satisfying $10^d = c^d = 5$, i.e. $d = \frac{ln\ 5}{ln\ 10} \approx 0.69897$. Note that $0 < d < 1$ as expected.

13. Projective plane geometry allows for duality, since the dual of a projective plane is again a projective plane (whereas the dual of an affine plane is not an affine plane). Typically one theorem in projective plane geometry represents many theorems in affine plane geometry. In the affine plane setting, theorems are often stated with exceptional cases to allow for lines that do not meet; moreover, in the affine setting, the lack of duality typically requires care in formulating the dual theorem. Similarly, in the projective setting the proofs themselves are simplified since one typically has fewer cases to consider. Projective plane geometry also simplifies the description of conics: over the real field or a finite field, there is projectively just one conic up to linear change of coordinates, unlike in the affine setting where ellipses, parabolas and hyperbolas must all be considered separately.



14. (a) This is the Theorem of Pappus. Although the figure has been drawn differently from what appears here, it is really the same as the figure shown on the right.

    (b) H is $(2,6)$; I is $(0,2)$. Lines have equations as follows:

ABC is $y = x$	AFH is $x = 2$	CDH is $y = 3x$
DEF is $y = 4x + 4$	BDG is $x = 3$	CEI is $x = 0$
AEG is $y = 1 - x$	BFI is $y = 5x + 2$	GHI is $y = 2x + 2$

15. Take as lines the exponential curves $y = ce^{kx}$ where $c, k \in \mathbb{R}$; together with vertical lines of the form $x = a$ where $a \in \mathbb{R}$. This makes the upper half-plane $y > 0$ into an affine plane. The axioms are easily checked directly. But even this should be obvious since what we have constructed is isomorphic to the classical real affine plane (i.e. the Euclidean plane). Indeed, the map $(x, y) \mapsto (x, e^y)$ is an isomorphism from the classical real affine plane $\mathbb{R}^2$ to the plane that we have just constructed.

In Section A you were not expected to provide explanation; however, I attach some comments below for your benefit.

1. Many statements (such as the Theorem of Pappus, or the statement that some line has exactly three points) cannot be either proved or disproved from the axioms—there are planes where they hold, and there are planes where they fail.
2. The projective plane of order 2 exists. No contradiction can be derived from the axioms, otherwise no projective planes (in particular the plane of order 2) would exist. One can also substitute the real projective plane in place of the plane of order 2 here, supplying a proof of relative consistency (i.e. assuming the real number system exists and satisfies the axioms for a

field, then so does the real projective plane); the advantage of using the plane of order 2, however, is that with only a finite number of points and lines, it is much easier to verify that it satisfies the axioms.

3. The Euclidean plane is in fact the classical affine plane coordinatized by the field of real numbers.
4. This statement misses the mark on many levels. First of all, the Euclidean plane is only one of many possible planes; its significance is due largely to tradition. It does not represent the true geometry of planes in our physical universe; it is merely a good approximation for many practical purposes. Finally, Euclid's axioms for this plane are somewhat lacking in precision, and have been replaced by modern sets of axioms.
5. As discussed in class and the handouts. In any case this is not hard to prove directly from the axioms.
6. This cannot be true, because there are nonclassical affine planes where the Theorem of Pappus fails. The smallest such nonclassical plane has order 9.
7. Recall that the Euclidean plane is itself an affine plane.
8. The axioms of plane geometry treat a 'point' as an undefined concept. It is only when defining a particular model (i.e. 'example') of the axioms that we provide an interpretation of 'point', 'line' and 'incidence'.
9. The notion of perpendicularity (or more generally, of angle) is not a relevant notion for affine planes in general.
10. As emphasized in class (and on the first handout), Euclidean plane geometry harbors questions that are as difficult as any to be found in modern mathematics. This includes unsolved and unsolvable problems, as well as problems of a computational nature that are provably uncomputable (i.e. for which it is known that no algorithmic solution exists).