Solutions to Optional Retake of Test 1

Section A: True/False

1. F	2. T	3. T	4. F	5. T	6. T	7. F	8. F	9. F	10. T

In Section A you were not expected to provide explanation; however, here are some comments for your benefit.

1. The set $\mathbb{R} \cup \{\infty\}$ is not a field. If it were, then $1+\infty \in \mathbb{R} \cup \{\infty\}$. Clearly $1+\infty \notin \mathbb{R}$, otherwise $\infty = (1+\infty)-1 \in \mathbb{R}$; but $1+\infty \neq \infty$, otherwise $1 = \infty - \infty = 0$. This is a contradiction.
2. This is a very standard fact derivable from the axioms.
3. Also a standard fact.
4. Physical planes are non-Euclidean; and over larger distances, and in the vicinity of large masses (such as stars) the non-Euclidean properties of space become measurable. For example, the angle sum of a triangle is not equal to 2π.
5. If $n \geq 3$ then three collinear points lie on more than one plane; whereas if $n = 2$ or three given points are not collinear, they lie in a unique plane. Recall our discussion and demonstration of this fact for $F = \mathbb{F}_3$ and $n = 2,3,4$ using the deck of Set® cards: there we took three cards not forming a 'set' (i.e. not collinear, and thus rather forming a triangle) and found the unique plane containing that triple of cards.
6. This is the usual construction of the classical projective plane over the field F.
7. In the axiomatic approach to affine plane geometry, the term 'point' is undefined. The poetic expression 'that which has position but no size' is imprecise and ambiguous.
8. There are many finite affine planes (in fact infinitely many such examples).
9. In the axioms for plane geometry, 'point' and 'line' are undefined terms; and algebraic curves cannot be defined unless the plane is classical (i.e. coordinatized by a field).
10. In the card game Set®, cards and sets represent points and lines of the classical affine 4-dimensional space over $\mathbb{F}_3$.

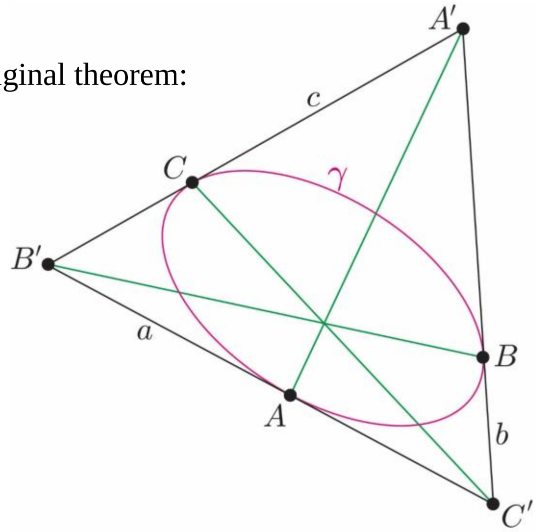
Sections B and C

11. The classical theorem of **Pappus** holds in a projective plane if and only if the plane is classical, meaning that it is coordinatized by a **field** . The proof of theorem is a somewhat involved exercise using **algebra** . The theorem makes no reference to the **order** of points on a line; nor does it require the notions of **distance** or **angle** which are featured prominently in Euclidean plane geometry.

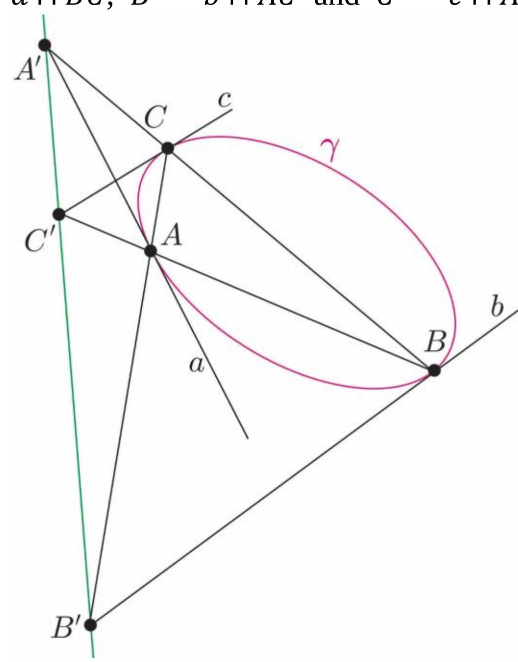
    We define a **quadrangle** to be a configuration of four **points** , no three of which are **collinear** . In the case of **affine** planes and **projective** planes, the existence of such a configuration is required by the **axioms** , in order to exclude degenerate examples. Also for such planes, any three distinct points are either **collinear** or they determine a unique **triangle** .

12. AB is $\begin{pmatrix}1\\0\\-1\end{pmatrix}$; AC is $\begin{pmatrix}1\\0\\0\end{pmatrix}$; BC is $\begin{pmatrix}1\\-1\\0\end{pmatrix}$; $A'B'$ is $\begin{pmatrix}3\\-1\\-5\end{pmatrix}$; $A'C'$ is $\begin{pmatrix}6\\-2\\-3\end{pmatrix}$; $B'C'$ is $\begin{pmatrix}2\\-3\\-1\end{pmatrix}$; P is $(1,-2,1)$; Q is $(0,3,-2)$; R is $(1,1,-1)$. We see that P, Q, R all lie on the line ℓ given by $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

13. Sketch of original theorem:



The dual may be stated as follows: Let a, b, c be lines tangent to an irreducible conic γ at distinct points A, B, C respectively. Then the three points $A' = a \cap BC$, $B' = b \cap AC$ and $C' = c \cap AB$ are collinear.



According to this particular choice of labels, the duality renames points and lines of the original theorem as $A \leftrightarrow a$, $B \leftrightarrow b$, $C \leftrightarrow c$. Moreover the old points A', B', C' correspond to the new lines BC, AC, AB respectively; whereas the new points A', B', C' correspond to the old lines AA', BB', CC' respectively. This is not the only possible naming convention; in fact we might have chosen to retain all the old labels in the new picture. But for once I have been more conventional in sticking to lower case labels for lines, and upper case labels for points. Although you were not asked to sketch the new theorem (the dual of the original statement), I have done so here (as on the right) for your benefit.

14. Scaling C by a factor of 3 in every direction gives a figure which is partitioned into five copies (distinguished by different colors in the figure shown on the right) of the original curve C. Thus C has Hausdorff dimension d satisfying $3^d = 5$, i.e. $d = \log_3 5 = \frac{\ln 5}{\ln 3} \approx 1.46497$. Note that $1 < d < 2$ as expected for a fractal curve in the plane.