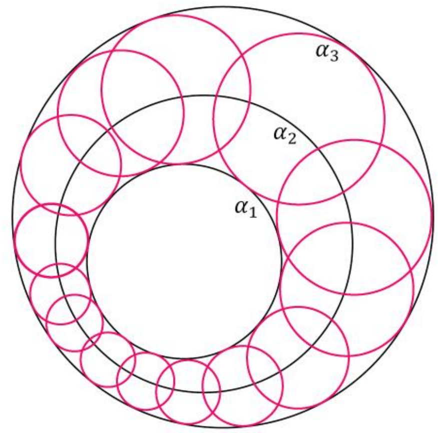
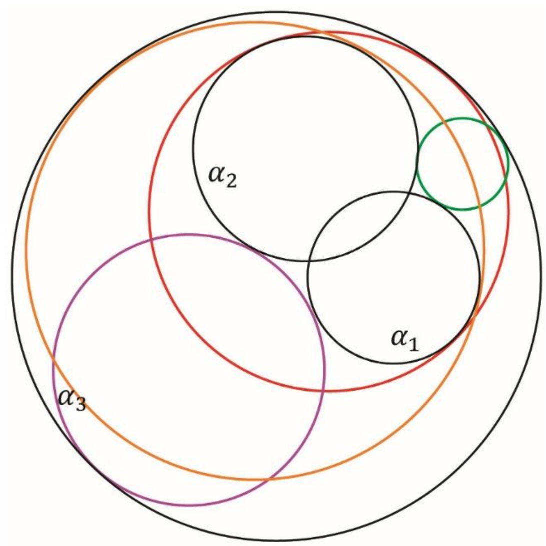
Solutions to Final Exam

1. (a) No such circles can occur because any circle tangent to both α_1 and α_3 would have to cross α_2 twice, as do any of the red circles shown on the right; so it could not be tangent to α_2.

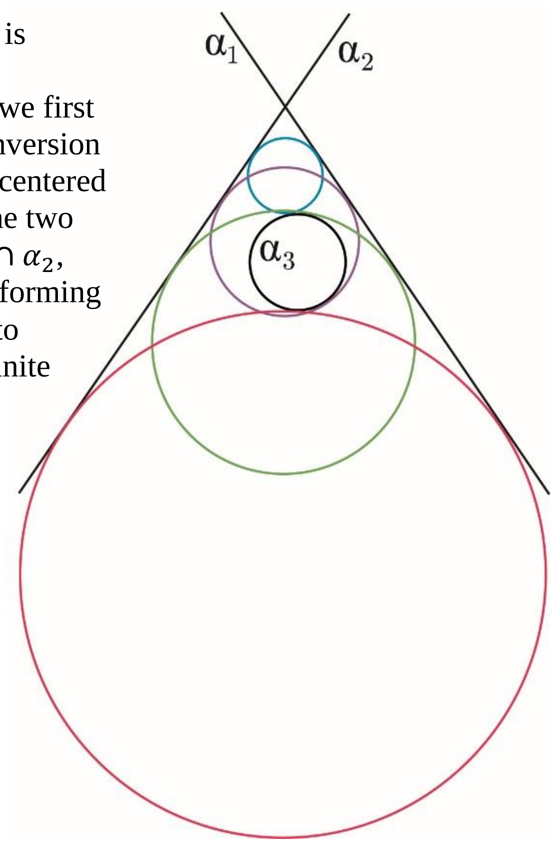
   Recall that in our analysis of Steiner chains of circles, we described a similar setting, in which the circles α_1 and α_3 were transformed to a pair of concentric circles, so that the red circles are permuted by rotational symmetry about the common center.



   (b) There are four such circles, as shown (in four randomly chosen colors).

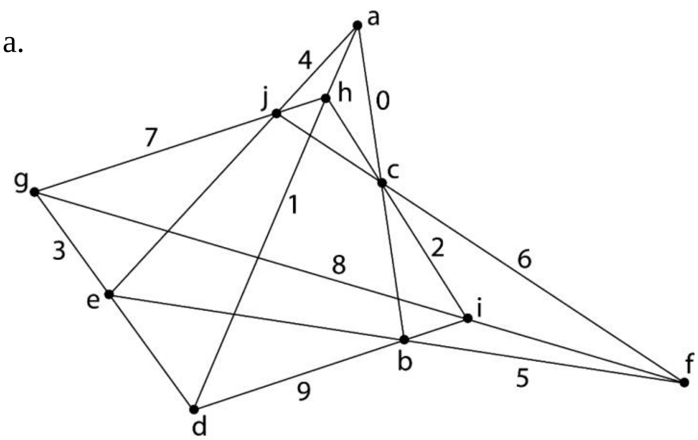


   The problem is conceptually simplified if we first perform an inversion in any circle centered at either of the two points of $\alpha_1 \cap \alpha_2$, thereby transforming α_1 and α_2 into circles of infinite radius.

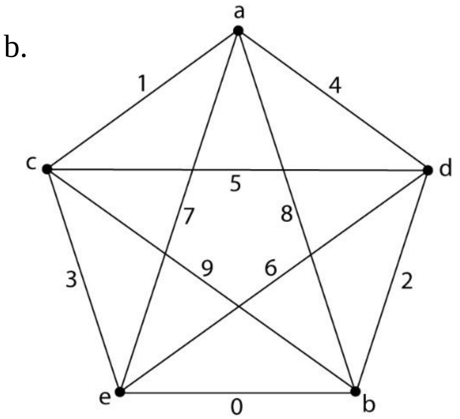


2. The conic has equation $ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$, where the constant coefficients $a, b, c, d, e, f \in \mathbb{R}$ are to be determined by the five known points, giving five linear equations $a = b = c = 4a + 4b + c + 4d - 2e - 2f = 4a + b + 4c - 2d + 4e - 2f = 0$ in six unknowns. Up to scalar multiple λ, the solution $(a, b, c, d, e, f) = \lambda(0,0,0,1,1,1)$ is unique. This uniquely determines the conic as $xy + xz + yz = 0$.

3. a.



b.



4. a. Since the sphere has positive curvature, we expect the angle sum of each triangle to exceed $\pi = 180°$.

   b. The angle sum of each triangle is $\frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{2} = \frac{7\pi}{6}$. One reads off the angles from the picture using the fact that all triangles are congruent, and they meet either 4 at a vertex or 6 at a vertex, with the angle sum at each vertex being 2π.

   c. The area of each triangle equals its angular excess $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$.

   d. Since the sphere has total area 4π, and each triangle has area $\frac{\pi}{6}$, the number of triangles is $4\pi/\frac{\pi}{6} = 24$.

5. Four classical types of plane geometry (affine, projective, hyperbolic and inversive) may be described either axiomatically, or constructed using coordinates taken from the **field** of real numbers for the relevant objects (points, lines, circles, etc.). For example, the Euclidean plane is the classical **affine** plane constructed using coordinates in $\mathbb{R}$. Historically, the Euclidean plane was viewed as the single correct or true plane geometry. We now recognize, however, that no one mathematical description perfectly captures the nature of physical reality. Moreover, all these plane geometries are relatively **consistent** (in the sense that no contradiction can possibly arise from studying one of them, unless a contradiction also arises from studying the others).

   For example, based on the Euclidean plane with point set $\mathbb{R}^2$, one may add a single point denoted by ∞ to obtain a representation of the **inversive** plane. Likewise, within the inversive plane, starting with an arbitrary choice of **circle** γ and taking circles orthogonal to γ, one constructs a copy of the **hyperbolic** plane. Or starting with the Euclidean plane, one may extend to the real **projective** plane by adding a set of new points, one for each **parallel** class of lines; then also joining up the new points using one new **line**. If there is a contradiction to be found in any one of these four geometries, the same logical contradiction would necessarily propagate to each of the other geometries. This argument is used to show relative **consistency** in the sense described above.

6. Each kite-shaped quadrilateral has angles 60°, 90°, 90°, 90° for an angle sum $\frac{\pi}{3} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = \frac{11\pi}{6}$. Its area (the same as the area of each fish) equals the angular defect $2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$.

7. Given any two distinct points P and Q, there are exactly $n+1$ circles passing through both P and Q. (If m is the number of circles passing through both P and Q then we consider the pairs (R,γ) where γ is a circle passing through P and Q, and R is another point on γ. Counting in two different ways, the number of such pairs is $n^2-1=(n-1)m$. We solve this to obtain $m=n+1$.)

If P is any point on a circle γ, then there are exactly $n-1$ circles tangent to γ at P. (Let m be the number of circles tangent to γ at P. Count in two different ways the number of pairs (Q,δ) where Q is a point not on γ and δ is a circle through P and Q tangent to γ. This gives $nm=n^2-n$. Solve to obtain $m=n-1$.)

If P is any point *not* on a circle γ, then there are exactly $n+1$ circles through P tangent to γ. (There are $n+1$ choices of point Q on γ; and for each Q, there is a unique circle through P tangent to γ at P.)

Given any circle γ, the number of circles α such that $|\alpha\cap\gamma|=2$ is $\frac{1}{2}n^2(n+1)$. (There are $\binom{n+1}{2}=\frac{1}{2}n(n+1)$ pairs of distinct points P,Q on γ. Each such pair gives n choices of α.)

Given any circle γ, the number of circles α such that $|\alpha\cap\gamma|=1$ is n^2-1. (There are $n+1$ choices of point P on γ; and for each P there are $n-1$ circles tangent to γ at P.)

Given any circle γ, the number of circles α such that $|\alpha\cap\gamma|=0$ is $\frac{1}{2}n(n-1)(n-2)$. (There are n^3+n-1 circles distinct from γ. Excluding those which meet γ in 1 or 2 points gives $n^3+n-1-\frac{1}{2}n^2(n+1)-(n^2-1)=\frac{1}{2}n(n-1)(n-2)$.)

8.

a. T	b. F	c. F	d. F	e. T	f. F	g. T	h. T	i. T	j. T

Comments in #8:

a. As described in class.

b. From the beginning of the semester, and as stated in the first handout, there is no algorithm which decides, given an arbitrary set of polygonal plane tiles, whether or not the Euclidean plane can be tiled using tiles of the given shape. (The problem is not just that no such algorithm is known. Rather, it is known that no such algorithm exists.)

c. A polarity maps points to lines and vice versa; it does not map points to points.

d. For general n, it is not known how many balls can be made to touch a given ball of the same size. This number is 2,6,12 for $n=1,2,3$; so the formula $3\times 2^{n-1}$ is correct for $n=2,3$ and incorrect for $n=1$. Also for $n=8$ the maximum number of balls touching a given ball is 240, somewhat fewer than 384 as predicted by the formula.

e. This property of Hausdorff dimension was pointed out in class (and used to check our computations in specific cases.)

f. The angle sum of a 30°-45°-90° triangle is $165° < 180°$, so no such triangle exists in the Euclidean plane.

g. The 1980's computer result of Clement Lam et. al. stated that no projective plane of order 10 exists. This implies that no affine or inversive plane of order 10 exists (since an inversive plane of order 10 would give rise to an affine plane of order 10; and an affine plane of order 10 would give rise to a projective plane of order 10).

h. This was described in class, with reference to p.362 of the book of Penrose.

i. Any line in the hyperbolic plane is topologically equivalent (i.e. homeomorphic) to a line in the Euclidean plane. 'Betweenness' for collinear points in the hyperbolic planes, is clearly seen from the Poincaré model, where lines are represented by arcs of circles.

j. Very small regular pentagons in the hyperbolic plane have angles only slightly less than 108°. By increasing the size of a pentagon, its angles decrease continuously, approaching 0° in the limit as the vertices tend to infinity. Any angle between 0° and 108° is realized at the vertices of some regular pentagon.