Overview of Modern Geometries

'Geometry' means literally 'measurement of the earth', and is the oldest branch of mathematics. Today we recognize many different branches of geometry, many of which do not involve measurement at all, as they do not possess any notion of distance or angle. Let's begin with a survey of some of the many types of geometry available for consideration, together with some comments about the most classical geometry of all (and the one that, by tradition, we have come to emphasize), the geometry of Euclid.

Geometries may be classified as ***discrete*** or ***continuous***. For example $\mathbb{R}^2$, (the Euclidean plane) is continuous; $\mathbb{Z}^2$ (the set of integer lattice points in the Euclidean plane) is discrete. A precise definition of a discrete space requires some topology; but intuitively, in a discrete space each point is separated from its nearest neighbours. Likewise a geometry may be ***finite*** (i.e. having only finitely many points or lines or other objects) or ***infinite***. For example the Euclidean plane is infinite. Every finite geometry is discrete, but not conversely (consider for example $\mathbb{Z}^2$ which is infinite but discrete).

Geometries may be classified by ***dimension***. We have 2D geometry (two-dimensional, or plane geometry); 3D geometry; 4D geometry; etc. We may be interested in geometry of any finite dimension (such as the 11 dimensions of modern string theory), or of infinite dimensions. There are also geometries of fractional dimension, or of negative dimension, or of no particular dimension.

Continuous plane geometries include the ***Euclidean plane*** (the high school plane geometry) and others such as the ***hyperbolic*** and ***elliptic*** planes. Similarly we have Euclidean, hyperbolic and elliptic geometries of every finite dimension. In particular hyperbolic 3-space is a good description of physical space in the vicinity of a large mass. (OK, this statement is an oversimplification…)

In every dimension we also have the notions of ***affine*** and ***projective geometry***. For example the Euclidean plane is simply the real affine plane (i.e. the affine plane constructed over the real numbers). The ***real projective plane*** may be viewed as an extension of the Euclidean plane in which we add 'points at infinity'. We can construct geometries over any field of numbers; for example if we replace the real number system $\mathbb{R}$ by the field $\mathbb{F}_2 = \{0,1\}$, we obtain an affine plane with four points and six lines (see Cederberg, pages 4 and 5), or a projective plane with 7 points and 7 lines (see Cederberg, page 11).

The basic objects of affine and projective geometry are ***points***, ***lines***, ***planes***, etc. In the inversive plane, the basic objects are ***points*** and ***circles***. In ***inversive 3-space*** the basic objects are ***points*** and ***spheres***.

In ***algebraic geometry*** one considers as basic objects all point sets that can be described by polynomial equations. Here the basic objects are points, curves, surfaces, etc. For example in the plane, this would mean all curves defined by polynomials; for example $x^3 + 3xy^2 = y^5$. In

analytic geometry one considers point sets defined by more general relations such as $x^3 \sin y = e^x$.

In ***differential geometry*** one considers a more general notion of ***distance*** (or ***metric***) than that of Euclidean space. This subject requires calculus to perform such routine tasks as determining ***geodesics*** (paths of shortest distance, at least locally). This is the language in which the theory of general relativity is written.

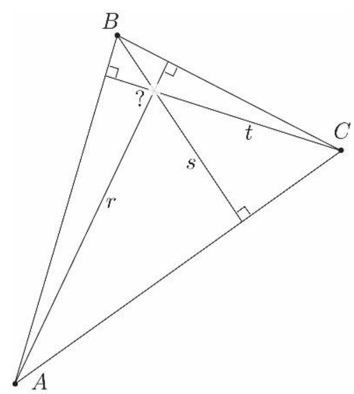
So there are geometries whose basic objects are points and lines only. There are geometries with points and lines and planes. And geometries with points and circles. Also geometries with points and curves. And so on. There are geometries without any points. (These are known as *pointless geometries*. No joke; but one wonders if those research pointless geometries have a hard time gaining respect, especially when searching for grant funding.)

No one geometry is 'correct' or 'incorrect'. Geometry is an abstract mathematical pursuit, often (but not always) motivated by concrete physical problems; in any case one must distinguish abstract geometry from any physical system that it may be intended to describe. An abstract description is at best an approximate or simplified (typically oversimplified) view of physical reality.

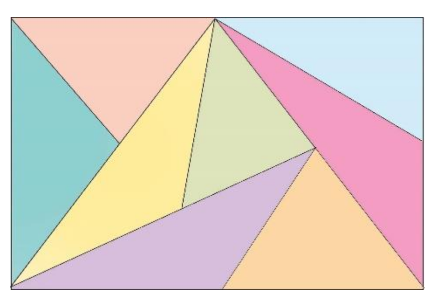
It is widely assumed that the Euclidean geometry of high school is the simplest of all geometries. This belief is misguided, however; Euclidean geometry abounds with subtleties and misconceptions. *Popular questions arising in Euclidean geometry range in level of difficulty from elementary, to difficult, to unsolved (or unsolvable)*. Here I offer illustrations to support this statement; and in many cases without justification—whenever the proof is beyond the scope of a course like this, I can provide references to anyone who is interested.

A Sample of Problems in Euclidean Plane Geometry

1. (*Not too hard*) A typical elementary problem in the Euclidean plane is to show that the three altitudes of a triangle intersect: given a triangle ABC, consider the three lines r, s and t through each vertex to the opposite side. Show that r, s and t are concurrent (i.e. they have a common point of intersection).



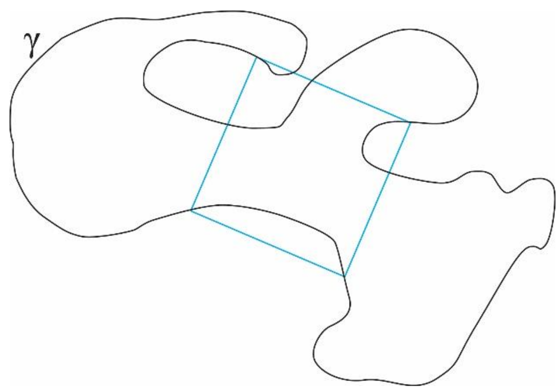
2. (*Hard*) There are many ways to decompose a rectangle into triangles of equal area; one such decomposition into 8 triangles of equal area is shown on the right. Show that it is *not possible* to decompose a given rectangle into an *odd* number of triangles of equal area.



3. (*Hard… and surprising… yet true!*) It is possible to decompose a closed ball of radius 1 in Euclidean space into five pieces which can be rearranged to form two balls of radius 1. This is a theorem of Banach and Tarski. (The original proof showed only that such a decomposition was possible using a finite number of pieces; later this was refined to five pieces.) (To explain the terminology, let me remind you that the *closed ball* of radius 1 centered at the origin in $\mathbb{R}^3$ is $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$.)

   Perhaps more surprising is the fact (also a theorem) that the analogous result in two dimensions is false: given a closed disk of radius 1 in the Euclidean plane, it is *not possible* to decompose the disk into a finite number of pieces which can then be rearranged to form two disks, each of the same size as the original.

4. (*Unsolved*) Let γ be a simple closed curve in the Euclidean plane (so γ does not cross itself; it can be drawn 'without lifting one's pencil', i.e. it has no jumps). Do there exist four points on γ which form the vertices of a square?



5. (*Incomputable in general*) Given a finite set of 'tiles' (called prototiles), assume that you have available an unlimited number of tiles of the same shape as the given prototiles. Is it possible to tile the Euclidean plane using the resulting tiles? In some cases the answer is easy, e.g. given a square as a prototile, the answer is 'yes'; given a circular disk as a prototile, the answer is 'no'. In some cases the answer is difficult; there exist certain sets of prototiles that allow a tiling of a large region of the entire plane, but not the entire plane. One might ask for an algorithm which, given an arbitrary set of prototiles as input, decides whether or not a tiling of the Euclidean plane is possible. However, it has been proved that there is no such algorithm.

Example #3 above (the Banach-Tarski Theorem) underscores my earlier claim that Euclidean geometry is strange in ways that may at first come as surprising. Of course the decomposition of a ball into five pieces is not possible to perform on material objects. (Otherwise we might be able to decompose a ball of gold and reassemble it into two balls, both as large as the original. This suggests a get-rich-quick scheme.) The Banach-Tarski Theorem is often referred to as a paradox; yet there is no contradiction to be found in it. It does not violate conservation of mass, because it does not apply to physical objects, because of the intricate nature of the pieces required in such a decomposition. The difficulty is much more fundamental than the graininess of physical matter (due to the finite size of individual molecules, atoms, and subatomic particles). In fact, the same confusion arises without considering physical matter at all: the theorem does not even apply to subsets of physical space, owing to the intricate nature of the subsets involved. These subsets are so intricate that they do not even have a well-defined volume; and so it does not contradict the familiar law of conservation of volume.

As ‘paradoxes’ go, the Banach-Tarski Theorem is not too far removed from another result, which is actually elementary to prove: It is possible to decompose a unit ball in $\mathbb{R}^3$ into infinitely many pieces (for example, take individual points) which can be reassembled to form two unit balls. This result is less impressive than the Banach-Tarski Theorem only because the decomposition requires an infinite number of pieces. Once again, this is impossible with physical objects; and conservation of volume is not violated because we are speaking of uncountably many pieces of volume zero, and uncountable sums are meaningless. (That’s why every infinite sum mentioned in Calculus II had only countably many terms.)

As evidence that the geometry of physical space is surprising, I point out that the triangles in our universe have angle sum not equal to 180°; the fact that the angle sum $\approx 180^\circ$ is a good approximation only over small distances, and the error in this approximation is measurable for sufficiently large triangles. This is a prediction of the general theory of relativity, and has been verified experimentally first in 1919, and (with greater accuracy) in 1959. In fact, it is meaningless to separate physical space from time: it is only spacetime that has absolute meaning, and this is further evidence that the Euclidean description can never be more than a good approximation. Moreover the geometry of physical space differs also from the ideal Euclidean description on the smallest scales, at distances of approximately 10^{-33}cm, where nothing is recognizable—even linear ordering of points. To summarize, (i) Euclidean space is bizarre; (ii) the physical space(time) we live in is bizarre; and (iii) these two spaces (Euclidean space and the space we live in) are bizarre in very different ways.

The difficulties we mention here arise primarily because the Euclidean plane is infinite (it has infinitely many points and infinitely many lines). Many of the difficulties arising in geometry can be avoided, in fact, by starting with geometries that do not require infinitely many points.