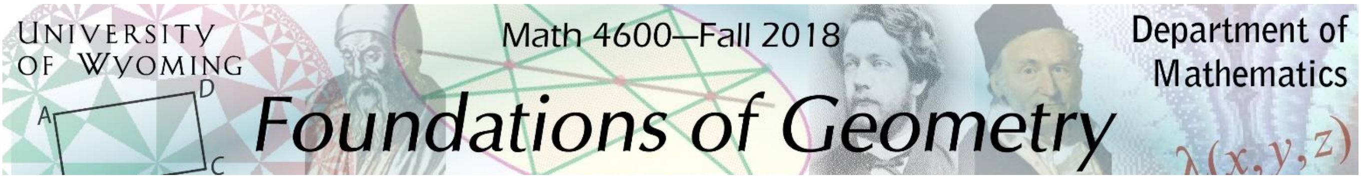


Another manifestation of the affine plane of order three:

Inflection Points on a Cubic Curve

A general polynomial $f(x,y) \in \mathbb{C}[x,y]$ of degree three defines a cubic curve in the plane $\mathbb{C}^2$ having nine inflection points, forming an affine plane of order 3 embedded in $\mathbb{C}^2$. By 'general' I mean generic: a 'randomly chosen' cubic curve has this property (although some specific choices of cubic curve will have degenerate behavior, resulting in fewer than nine inflection points). Let's observe this with a curve that is chosen carefully enough to avoid degeneracies, but still nice enough to be able to explicitly find the inflection points. For this purpose I choose the curve C defined by the cubic equation

$$y^3 + (y+1)^3 = x^3. \tag{1}$$

We use implicit differentiation to find y' and y'' as follows:

$$(2y^2 + 2y + 1)y' = x^2 \tag{2}$$

$$(4y+2)(y')^2 + (2y^2 + 2y + 1)y'' = 2x \tag{3}$$

The curve C consists of all points $(x,y) \in \mathbb{C}^2$ satisfying (1). Inflection points of the curve are those points which simultaneously satisfy two conditions: equation (1), and the condition $y'' = 0$. I will not present all the algebra required to simultaneously solve such a system of two polynomial equations in two unknowns; but if I show you nine solutions, you should have no trouble verifying that they are inflection points as claimed. Later in the course we will encounter a general principle which allows us to conclude that these are the only inflection points. Let $\omega \in \mathbb{C}$ be a root of $\omega^2 + \omega + 1 = 0$. Note that $\omega^3 = 1$ (i.e. ω is a cube root of unity). Then the nine inflection points are

$$(-\omega^2, -1) \quad (0, -\tfrac{1}{2}) \quad (\omega^2, 0)$$

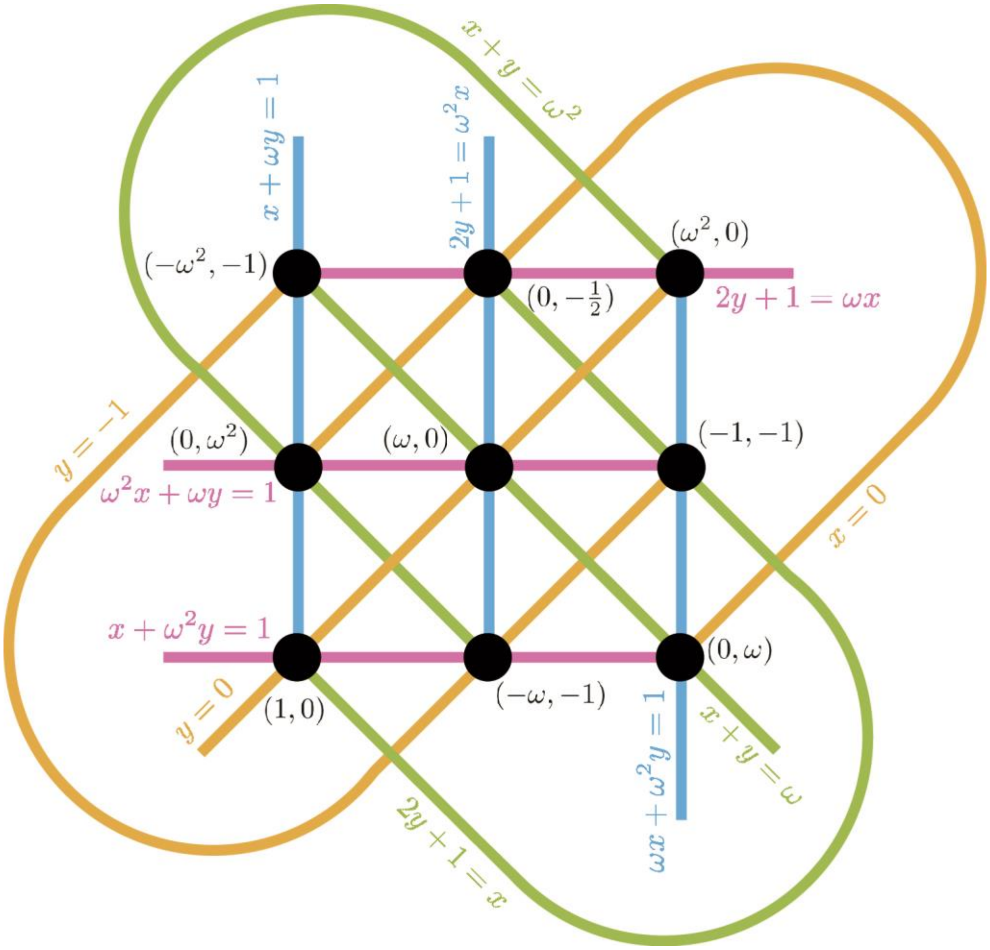
$$(0, \omega^2) \quad (\omega, 0) \quad (-1, -1)$$

$$(1,0) \quad (-\omega, -1) \quad (0, \omega)$$

and they are joined by the twelve lines

$x = 0$	$2y + 1 = x$	$x + \omega y = 1$	$x + \omega^2 y = 1$
$y = 0$	$2y + 1 = \omega x$	$x + y = \omega$	$x + \omega^2 y = \omega$
$x = -1$	$2y + 1 = \omega^2 x$	$x + y = \omega^2$	$x + \omega y = \omega^2$

thus forming an affine plane of order three embedded in $\mathbb{C}^2$:



To avoid further complicating this image, I have not tried to depict the curve C itself, just its nine inflection points and their joining lines. As usual, this looks a little clearer in the electronic document (on the course website) than in the grayscale hardcopy distributed in class.

The fact that some of the lines look curved in our Euclidean picture, is due to the fact that $\mathbb{R}$ does not contain a primitive cube root of unity. In its native environment, there is nothing 'curved' about any of the lines of the structure depicted above; but the Euclidean plane $\mathbb{R}^2$ is not endowed with enough features to faithfully convey the affine plane of order three. Many fields besides $\mathbb{C}$ do contain an element ω with the required properties; for example the field $\mathbb{F}_7 = \{0,1,2,3,4,5,6\}$ of order seven contains such an element $\omega = 2$, and so the classical affine plane of order seven does contain an affine subplane of order 3.