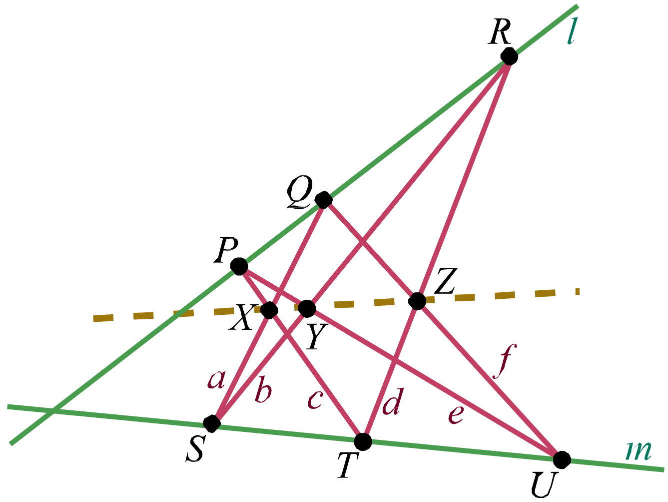
HW3 Due Tuesday, November 27, 2018

The following theorem is valid in all *classical* projective planes (i.e. those projective planes which can be coordinatized by a field.) There also exist nonclassical projective planes in which the Theorem of Pappus fails; thus the axioms for projective plane geometry are not complete.

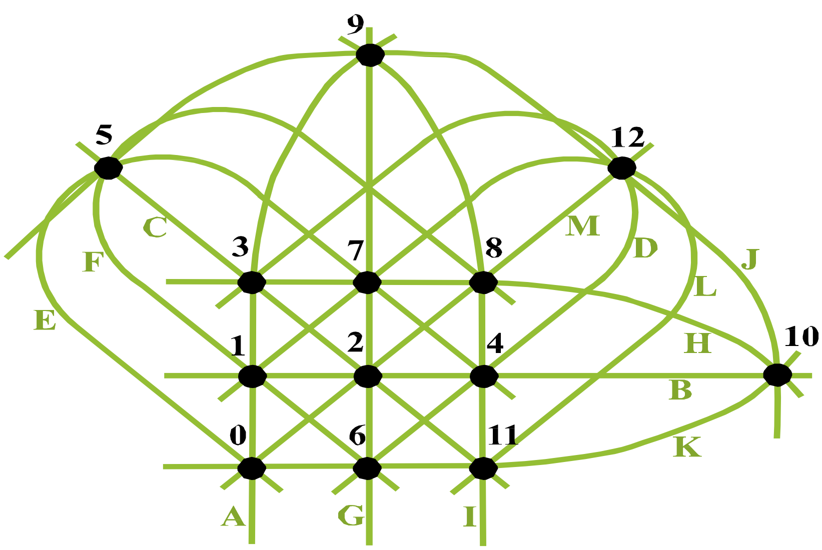
Pappus' Theorem. Let *l* and *m* be distinct lines. Let *P*, *Q*, *R* be distinct points on ℓ (not on *m*), and let *S*, *T*, *U* be distinct points on *m* (not on *l*). Let

- *a* be the line joining *Q* and *S*;
- *b* be the line joining *R* and *S*;
- *c* be the line joining *P* and *T*;
- *d* be the line joining *R* and *T*;
- *e* be the line joining *P* and *U*;
- *f* be the line joining *Q* and *U*;
- *X* the intersection of *a* and *c*;
- *Y* the intersection of *b* and *e*; *Z* the intersection of *d* and *f*.

Then *X*, *Y* and *Z* are collinear.



Now consider the projective plane of order 3, which we illustrate by the following picture:

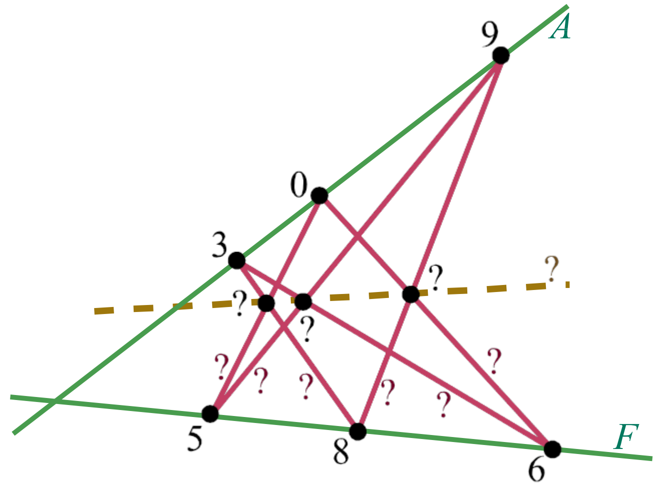


This plane has thirteen *points*: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 and thirteen *lines*:

$A = \{0,1,3,9\}$	$D = \{3,4,6,12\}$	$G = \{6,7,9,2\}$	$J = \{9,10,12,5\}$	$M = \{12,0,2,8\}$
$B = \{1,2,4,10\}$	$E = \{4,5,7,0\}$	$H = \{7,8,10,3\}$	$K = \{10,11,0,6\}$	(notice the
$C = \{2,3,5,11\}$	$F = \{5,6,8,1\}$	$I = \{8,9,11,4\}$	$L = \{11,12,1,7\}$	pattern here?)

1. Verify that the conclusion of Pappus' Theorem holds in the following instance: Consider the points 3, 0, 9, 5, 8, 6 in place of *P*, *Q*, …, *U* respectively; note that the lines ℓ and *m* become the lines *A* and *F* of the projective plane of order 3.

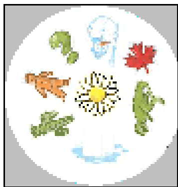
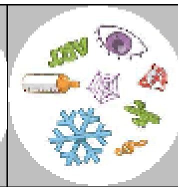
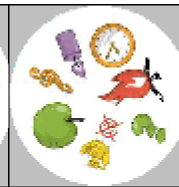
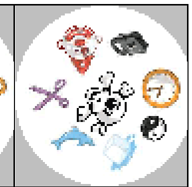
   Identify the lines and points that are labeled by question marks in the figure at the right, and verify that the three points of intersection are indeed collinear. What is the line joining them? (shown as a dotted line in our figure).



If *S* is any conic in the plane, every line ℓ meets the conic in 0, 1 or 2 points, and then ℓ is called a *passant*, *tangent*, or *secant* line respectively. Any point *P* of the plane lies on 0, 1 or 2 tangents, in which case *P* is called an *interior point*, a *point of the conic* (also called an *absolute point*), or an *exterior point* respectively. It may be shown that the point set *S* = {0, 1, 2, 6} is a conic in the projective plane of order 3 shown above. (Please assume this; don't attempt to prove it.)

2. For the conic *S* = {0, 1, 2, 6} in the projective plane of order 3 shown above:
   a. List the passant lines.
   b. List the tangent lines.
   c. List the secant lines.
   d. List the interior points.
   e. List the absolute points (i.e. points of the conic).
   f. List the exterior points.
   g. Complete the blanks in the following sentences with the correct numbers:
      There are ____ passant lines, each of which passes through
      ____ interior points, ____ absolute points, and ____ exterior points
      There are ____ tangent lines, each of which passes through
      ____ interior points, ____ absolute points, and ____ exterior points.
      There are ____ secant lines, each of which passes through
      ____ interior points, ____ absolute points, and ____ exterior points.
   h. Complete the blanks in the following sentences with the correct numbers:
      There are ____ interior points, each of which lies on
      ____ passant lines, ____ tangent lines, and ____ secant lines.
      There are ____ absolute points, each of which lies on
      ____ passant lines, ____ tangent lines, and ____ secant lines.
      There are ____exterior points, each of which lies on
      ____ passant lines, ____ tangent lines, and ____ secant lines.
   i. Compare your answers in (g) and (h). How does this illustrate the principle of duality?

3. Consider the cubic curve given by the equation $y^2z = x^3 - xz^2$ in projective coordinates. In the affine plane $z = 1$ its equation becomes $y^2 = x^3 - x$. Over the field $\mathbb{F}_7 = \{0,1,2,3,4,5,6\}$, the curve has seven affine points $(0,0), (1,0), (6,0), (5, \pm1), (4, \pm2)$; i.e. in projective coordinates, the seven points $(0,0,1), (1,0,1), (6,0,1), (5, \pm1,1), (4, \pm2,1)$; also the 'vertical' point $(0,1,0)$ at infinity, which we denote simply by O. So altogether, the curve has eight points. These points are represented in the SpotIt® deck (see ericmoorhouse.org/pg27 and click on the picture of the SpotIt® deck) by the eight cards

							
$(0,0)$	$(1,0)$	$(6,0)$	$(5,1)$	$(5,-1)=(5,6)$	$(4,2)$	$(4,-2)=(4,5)$	O

respectively. Note that the curve is symmetric about the x-axis: the reflection in the x-axis is the map $(x,y,z)\mapsto(x,-y,z)$; it maps the curve to itself.

To add two points P,Q of the curve, assuming $P\neq Q$, consider first the secant line PQ. According to Bezout's Theorem, this secant line must intersect the curve in a third point R (counting multiplicity). Reflecting R in the x-axis gives the point $P+Q$. One must account for intersection multiplicity, however. Thus for example if $P=Q$ is a point of the curve, then in place of a secant line we must take the tangent line at P.

Example: To add the points $P=(5,1)$ and $Q=(6,0)$, note that the secant line PQ is given by $y=-x-1$ which meets the curve at a third point $(4,2)$. Reflecting this point in the x-axis gives the point $(4,5)$; so $P+Q=Q+P=(4,5)$. Alternatively, from the SpotIt® demonstration site, we see that

PQ is the line  which meets the curve at a third point $(4,2)=$ 

whose reflection in the x-axis is $(4,-2)=(4,5)$. Third method: one can instead use homogeneous coordinates for the projective plane: the line joining $P=(5,1,1)$ and $Q=(6,0,1)$ is $\begin{pmatrix}1\\1\\1\end{pmatrix}$; and the third point on the curve satisfying this equation $x+y+z=0$ is the point $(4,2,1)$. This reflects to the point $(4,-2,1)=(4,5,1)$.

Second Example: To add the points $P=(5,1)$ and $S=(1,0)$, note that the secant line PS is given by $y=2x+5$. We see that this secant line passes through no other points of the curve. But wait: by implicit differentiation, the tangent line at a point (x,y) of the curve $y^2=x^3-x$ has slope y' satisfying $2yy'=3x^2-1$, i.e. $y'=\frac{3x^2-1}{2y}$. So the tangent line at P has slope has slope 2; that is, the line PS is tangent to the curve at P. So P is really a double point of the curve. Reflecting P in the x-axis gives the point $(5,-1)=(5,6)$, whence $P+S=S+P=(5,6)$.

Third Example: To add the point $T=(1,0)$ to itself, we must take the tangent line at T. The formula for y' above gives the slope of the tangent line at T to be infinite; so it is the vertical line $x=1$. The third point of the curve on this line is the point $O=(0,1,0)$. Reflecting this in the x-axis gives $(0,-1,0)=(0,1,0)=O$ again; so $T+T=O$.

Fourth Example: To add the points $P=(5,1)$ and $P'=(5,6)$, note that the secant line PP' is the vertical line $x=5$ which passes also through O, the vertical point at infinity. As in the previous example, $P+P'=O$.

Fifth Example: To add the point $P=(5,1)$ to itself, recall (above) that the tangent line at P has slope 2; it is the line $y=2x+5$, meeting the curve in a third point $(1,0)$ which reflects to the point $(1,0)$. So $P+P=(1,0)$.

+	O	(0,0)	(1,0)	(6,0)	(5,1)	(5,6)	(4,2)	(4,5)
O								
(0,0)								
(1,0)			O		(5,6)			
(6,0)					(4,5)			
(5,1)			(5,6)	(4,5)	(1,0)	O		
(5,6)					O			
(4,2)								
(4,5)								

These five examples have been included in the addition table above for the group of eight points of the curve. *Complete the table by correctly entering the missing values*.

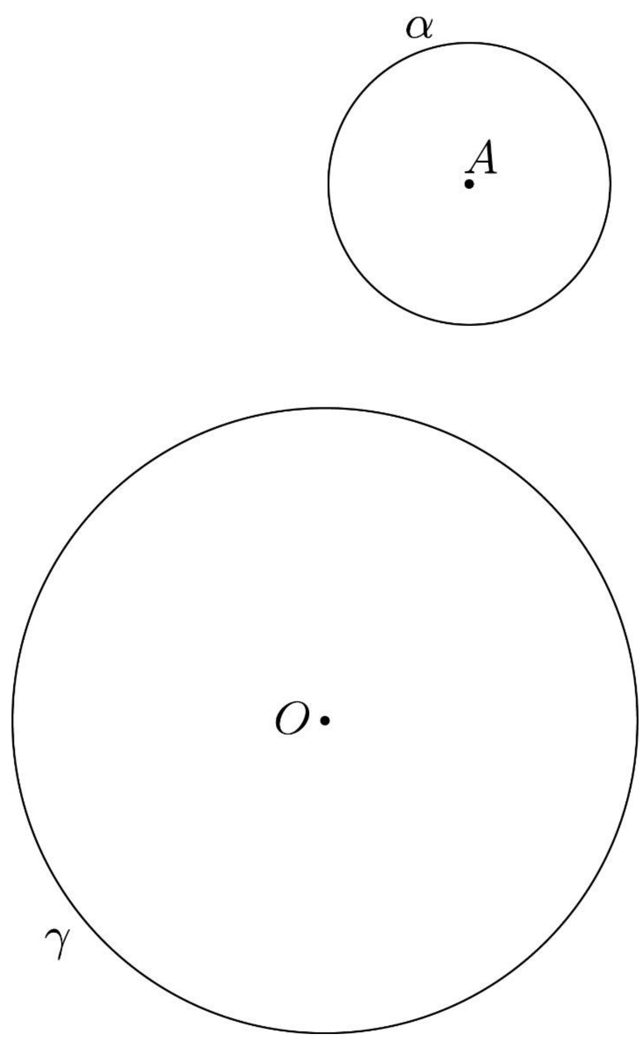
Remarks: You should quickly confirm that O is the additive identity element for this group: $A + O = A$ for every point A on the curve. And while the geometric rules given above suffice to complete the addition table, you may also find it useful to use the associative law. (As mentioned in class, it may be shown that addition of points is associative—this is not obvious!) For example,

$$(5,6) + (5,6) = (5,6) + [(5,1) + (1,0)] = [(5,6) + (5,1)] + (1,0) = O + (1,0) = (1,0)$$

by the previous remark.

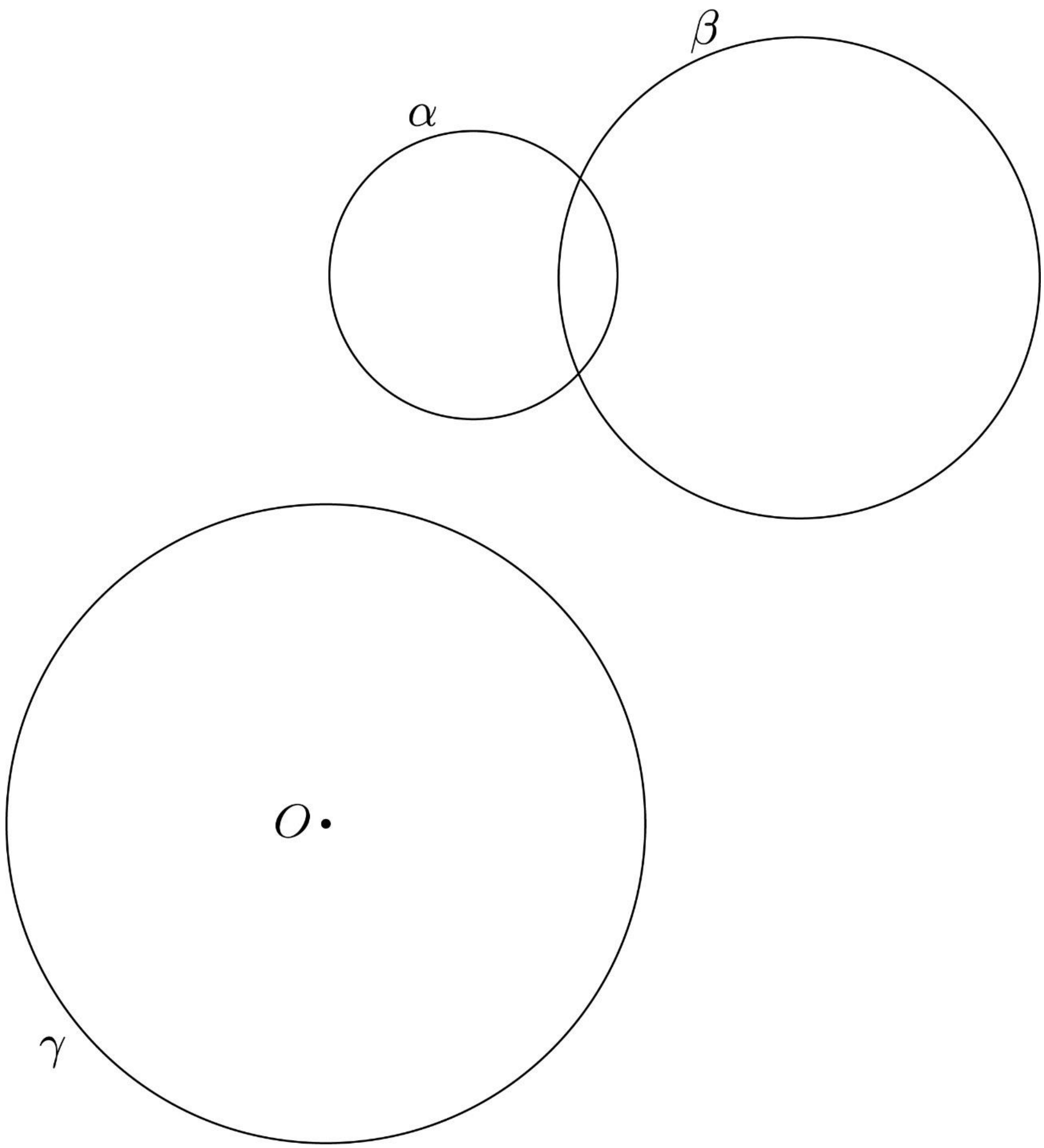
The remaining problems below require straightedge-and-compass constructions. Perform all constructions in the space provided. Do not erase any of your work; show clearly all line segments and circular arcs used in your construction. Clearly label the final points and/or lines required by each question. If you mess up and need to start again, print a fresh copy of this assignment from the course website. For demonstrations of the more basic constructions, follow the link provided on the course website.

4. The circle α is centered at A as shown. Construct the image $\alpha'$of α under inversion in γ, a circle centered at O.



Also construct A', the inverse of A in γ. Is A' the center of α'? Explain.

5. Construct the inverses α', β' (respectively) of the circles α, β (as shown) in the circle γ (centered at O).



Measure (as well as you can using a protractor) the angle between circles α and β. (This requires first drawing tangent lines to α and β at a point of intersection.) Do the same for α' and β'.

angle between α and β =

angle between α' and β' =

How do these two angles compare?