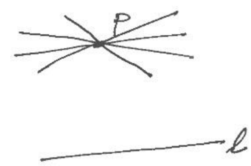
Solutions to HW2

Examples 1–5,7,8 are *not* affine planes; examples 6,9,10 *are* affine planes. By now it should be clear that one does not stumble upon affine planes easily or by chance; only very carefully designed structures will satisfy the governing axioms. Here are some explanations in each case:

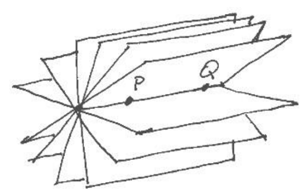
1. This does not even satisfy A1; for example, the 'points' Texas and Tennessee lie on more than more than one 'line' (T, E and S).

2. Once again, A1 fails. This time, the 'points' T and E lie on more than one 'line' (Texas, Tennessee, Connecticut, Kentucky, Massachusetts, Minnesota, Vermont, and West Virginia).

3. Although A1 and A3 hold, A2 fails. Given any line ℓ and point P not on ℓ, there are infinitely many lines through P not meeting ℓ.

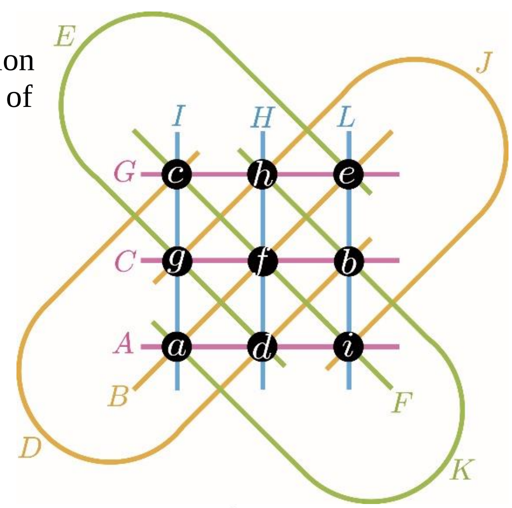


4. Axiom A1 fails. For any two 'points' $P \neq Q$, there are *infinitely many* 'lines' containing both P and Q.

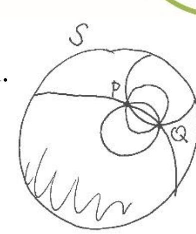


5. A1 fails since 'points' Bob and Dave are not on the same 'line'.

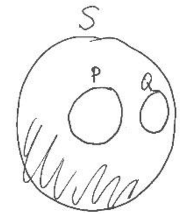
6. This is an affine plane of order 3. From the illustration shown, you can see that it is isomorphic to the plane of order 3 presented in class previously.



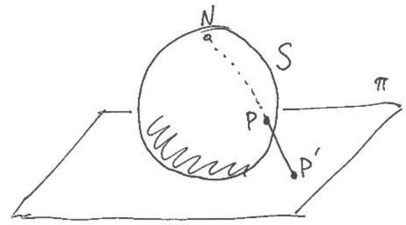
7. A1 fails since two 'points' $P \neq Q$ are on many 'lines', as shown.



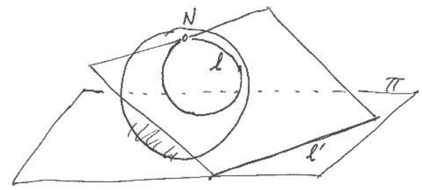
8. A1 fails since two ‘points’ $P \neq Q$ may be on no ‘lines’ (as shown), or two.



9. This is in fact isomorphic to the real affine plane (i.e. the Euclidean plane). This is most easily seen by stereographic projection, as follows. Denote by N the north pole of S; recall that this is not a ‘point’ of our plane. Consider the plane π tangent to S at the south pole, so that S sits on π as shown. For each ‘point’ (i.e. point of S other than N), the ray NP intersects π at a point $P' = \pi \cap NP$, and we have a bijection $P \leftrightarrow P'$ between ‘points’ of S and ordinary points of the Euclidean plane S.



Under this bijection, ‘lines’ of S (ordinary circles ℓ of S, with N removed) correspond to Euclidean lines of S (illustrated by the correspondence $\ell \leftrightarrow \ell'$, shown).



10. Once again, this is isomorphic to the classical real affine plane (i.e. the Euclidean plane $\mathbb{R}^2$). The ‘point’ (x, y) lies on the ‘line’ $\{(x, x^2 + bx + c) : x \in \mathbb{R}\}$ iff the usual point $(x, y - x^2)$ lies on the usual Euclidean line $\{(x, bx + c) : x \in \mathbb{R}\}$; so renaming points via $(x, y) \longmapsto (x, y - x^2)$ and lines via $\{(x, x^2 + bx + c) : x \in \mathbb{R}\} \longmapsto \{(x, bx + c) : x \in \mathbb{R}\}$ gives an isomorphism from this new plane to the Euclidean plane.