HW2 Due Thursday, October 11, 2018

Instructions: In this assignment, you are given a variety of new point-line incidence structures. I have introduced 'points' and 'lines' of each new structure using quotation marks, in order to distinguish them from the ordinary points and lines of Euclidean geometry. In each case, indicate whether or not the specified structure is an affine plane. Justify your answers, indicating (whenever the structure is not an affine plane) at least one way in which it fails to satisfy the axioms of affine plane geometry.

1. Take fifty 'points' consisting of the fifty states of the USA. Take twenty-six 'lines' to be the letters of the Roman alphabet. Incidence of a 'point' P on a 'line' ℓ means that the name of the state P contains the letter ℓ; for example the 'point' Wyoming lies on the 'line' M but not on the 'line' B. Does this give an affine plane? Explain.

2. In Question 1, reverse the roles of points and lines to obtain an incidence structure with twenty-six 'points' (letters of the alphabet) and fifty 'lines' (states). Incidence is as before, but reversed; for example, the 'point' B lies on the 'line' Alabama but not on the 'line' New York. Is this new structure an affine plane? Explain.

3. Euclidean 3-space $\mathbb{R}^3$ contains ordinary points, lines and planes. Construct a new geometry whose 'points' are the ordinary points of $\mathbb{R}^3$, and whose 'lines' are the ordinary lines of $\mathbb{R}^3$. Incidence is ordinary geometric containment. Is this an affine plane? Explain.

4. Varying the structure of Question 3, construct a new geometry whose 'points' are the ordinary points of $\mathbb{R}^3$, and whose 'lines' are the ordinary *planes* of $\mathbb{R}^3$. Incidence is the usual geometric containment. Is this an affine plane? Explain.

5. The Mariner's Club has twelve members, and nine committees:
   a. the Board of Governors: Alice, Bob, Ian and Karen;
   b. the Board of Trustees: Carol, Dave, Karen and Liz;
   c. the Properties Committee: Dave, Fred, George and Ian;
   d. the Social Committee: Alice, Dave, Eve and Harry;
   e. the Treasury Board: Bob, Eve, George and Liz;
   f. the Constitutional Revision Committee: Bob, Carol, Fred and Harry;
   g. the Fundraising Committee: Carol, Eve, Ian and Janet;
   h. the Recruitment Task Force: George, Harry, Janet and Karen; and
   i. the Committee on Committees: Alice, Fred, Janet and Liz.

   Take 'points' to mean individual club members; and 'lines' to mean committees. Incidence is the natural relation of committee membership. Is this an affine plane? Explain.

6. Consider the Mariner's Club of Question 5, but reverse the roles of points and lines: take 'points' to mean committees, and 'lines' to mean individual members. Incidence is again the natural relation of committee membership. Does this give an affine plane? Explain.

7. Consider a *Euclidean sphere* S (not a *ball*; a sphere is the surface boundary of a ball, e.g. $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$). Note that every plane of Euclidean 3-space intersects S either in a point, or a circle, or is disjoint from S; and every circle on S arises in this way, i.e. as a plane intersection.

   Now take 'points' to be just the ordinary points of S, and 'lines' to be the ordinary circles in S. Incidence is the natural one, i.e. geometric containment. Does this form an affine plane? Explain.

8. Consider the structure of Question 7, but reverse the roles of 'points' and 'lines': Take 'points' to be the ordinary circles of S; and take 'lines' to be the ordinary points of S. Is this an affine plane? Explain.

9. As in Question 7, we fix a Euclidean sphere S; and now fix a point on S which we call the north pole. Take as 'points', the ordinary points of *S except for the north pole*; and take 'lines' to be the ordinary circles of S passing through the north pole. As before, incidence is the usual containment. Does this form an affine plane? Explain.

10. From the Euclidean plane $\mathbb{R}^2$, we construct a new incidence structure as follows: 'Points' are the usual points $(x, y) \in \mathbb{R}^2$. 'Lines' are of two types:
    - Each pair of real numbers b, c gives an ordinary parabola $\{(x, x^2 + bx + c) : x \in \mathbb{R}\}$. Ordinary parabolas of this form are taken to be 'lines' of our new geometry.
    - Each real number a gives an ordinary vertical line of the form $\{(a, y) : y \in \mathbb{R}\}$. These ordinary lines are also taken to be 'lines' of our new geometry.

    Is this new point-line incidence structure an affine plane? Explain.

The great mathematician David Hilbert said that in axiomatic geometry, we should at any time be ready to replace the names 'point', 'line', and 'plane' by 'table', 'chair', and 'beer mug' … or words to that effect, in German. It is hoped that these exercises will help you transcend the usual attachments of our common words 'point', 'line' and 'incidence'.

David Hilbert
1862–1943