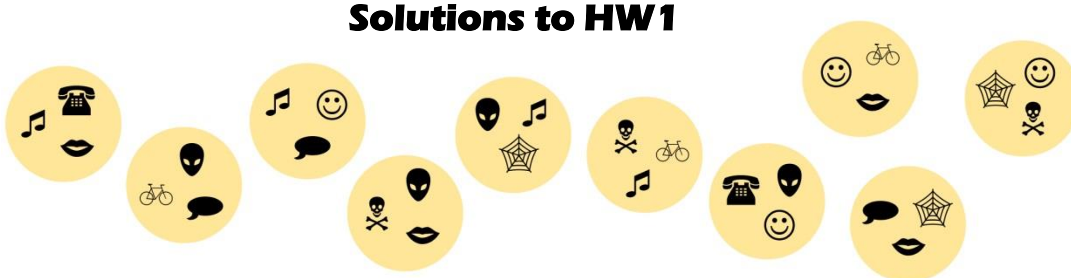
Solutions to HW1



1. In an affine plane of order 3, every point is on 4 lines. Simple counting shows that each of the points 🚲, 💬, 🕸, and ☠ is missing a line through it; and the point ☎ is missing two lines through it. So three possibilities for the missing lines are

   a.  or b.  or c. 

   Cases (a) and (b) are not possible since the point ☠ is already joined to both 🚲 and 🕸. So the missing two lines must be those listed in case (c).

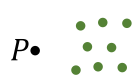
2. (a) The four points 🚲, 💬, ☎, and ♬ form a quadrangle in the plane **A**.

   (b) Define an *ordered quadrangle* to be a *sequence* (P, Q, R, S) of four points, no three collinear. Such a sequence gives rise to a quadrangle $\{P, Q, R, S\}$, this being a *set* of four points (so that the order does not matter). Note that every quadrangle gives rise to $4! = 24$ ordered quadrangles, distinguished by the 24 distinct ways of ordering the four points. So the number of ordered quadrangles is $24n$ where n is the number of quadrangles. On the other hand, there are $9 \times 8 \times 6 \times 3 = 1296$ distinct ordered quadrangles (P, Q, R, S). This is because

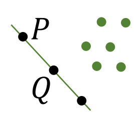
   there are 9 choices of P;

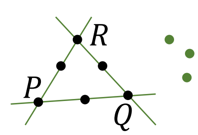
   for each P, there remain 8 choices of Q;

   given P and Q, there remain 6 choices for R; and

   given P, Q, R, there remain 3 choices for S.

   Solving $24n = 1296$ gives $n = 54$ quadrangles.

Alternatively (and less directly), the total number of 4-sets of points (i.e. sets consisting of 4 points) is $\binom{9}{4} = 126$. Every 4-set of points is either ‘good’ (a quadrangle), or ‘bad’ (a set of three points all on some line ℓ, plus a point not on ℓ). In the latter case, there are 12 choices for the line ℓ; and for each such line there are 6 choices of fourth point not on ℓ; so there are $12 \times 6 = 72$ ‘bad’ 4-sets of points. This leaves $126 - 72 = 54$ ‘good’ 4-sets of points (i.e. quadrangles).

The direct method (our first solution) is preferable since it immediately gives the number of quadrangles in any affine plane (try it).

3. The unique line joining the points 🚲 and ☎ is .

4. The unique line through the point ☠ which does not intersect the line  is .

5. (a) Assuming the first approximation is a unit square (of area 1 square unit), then the nth approximation has area $\left(\frac{5}{9}\right)^{n-1}$ which tends to 0 in the limit as $n \to \infty$. So our fractal, which is the limiting point set, has area zero.

   (b) One-fifth of the point set (highlighted in the accompanying figure by red shading), when scaled by a factor of 3, yields the entire original fractal. According to our formula, the Hausdorff dimension d of the fractal satisfies $3^d = 5$, so $d = \frac{\ln 5}{\ln 3} \approx 1.4650$. Note that this dimension is between 1 and 2 as we would expect based on geometric considerations.