Final Exam 1:15-3:15 pm Thursday, December 13, 2018

Instructions: Answer all questions in the space provided (or attach additional pages as needed). You are permitted to use pencils/pens, one "cheat sheet" (one side of an 8½″×11″ sheet with your own handwriting only), and a calculator. No sharing of calculators, or use of cellphones or audio listening devices, will be permitted. Show complete work and use complete sentences when required. Total value of questions: 100 points.

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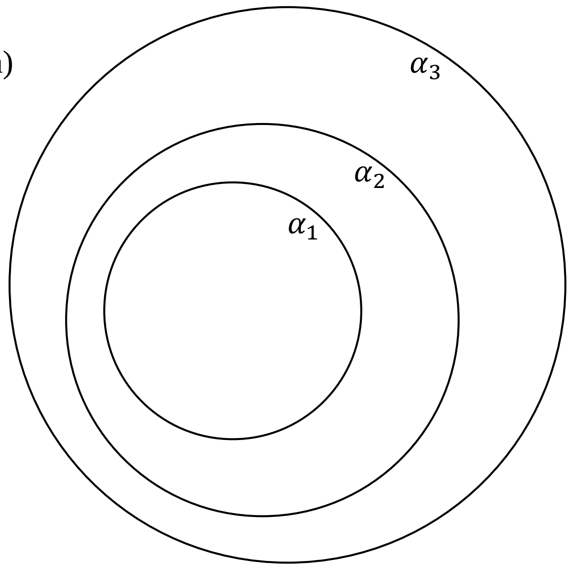
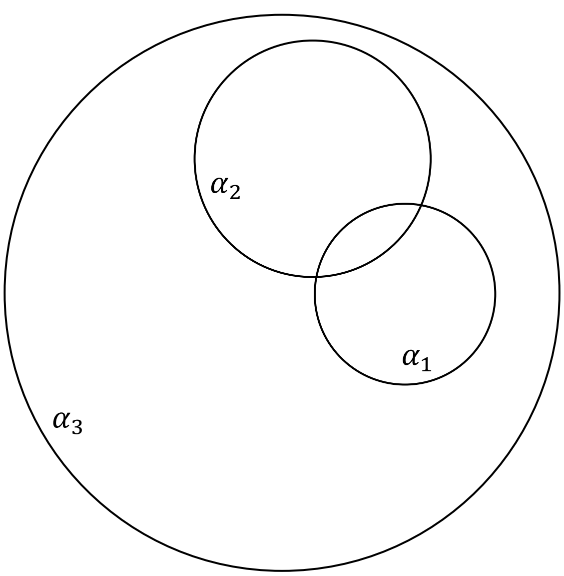
1. (*10 points*) In each of the figures below, indicate the *number* of circles γ which are *tangent* to all three of the circles α_1, α_2 and α_3. (*Hint*: In each case there is an inversion which simplifies the question.)

(a)



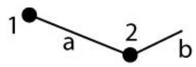
(b)



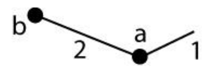
2. (*10 points*) Consider the five points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(2,2,-1)$, $(2,-1,2)$ of the real projective plane. (As customary with coordinates for points in a classical plane, we write $(1,0,0)$ as an abbreviation for the one-dimensional subspace of $\mathbb{R}^3$ spanned by $(1,0,0)$; and similarly with the other points.) Since no three of the five points given are collinear, there is a unique conic γ passing through all five points. Determine the equation of this conic.

3. (*10 points*) In each of the following, two configurations of points and lines are given. In each case, show that the second configuration is dual to the first by completing the missing labels for lines and points using the same names given for points and lines (respectively) in the original configuration, so as to indicate the duality. The first is done for you as an example.

(example)

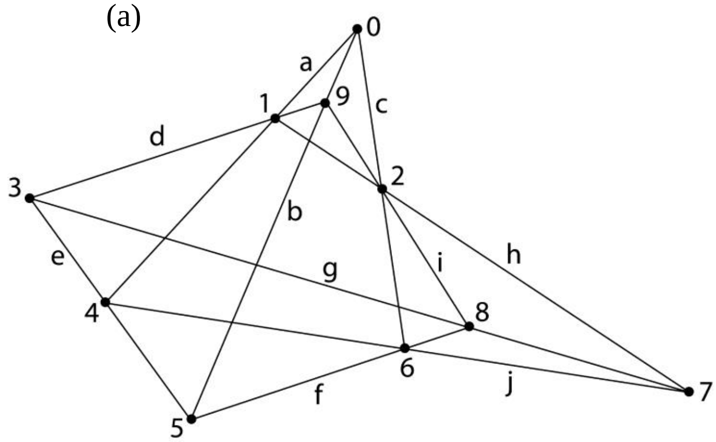


2 points, 2 lines (self-dual)

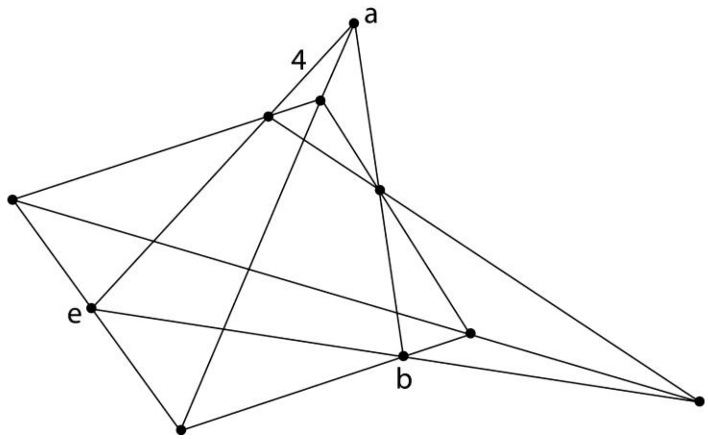


2 lines, 2 points

(a)

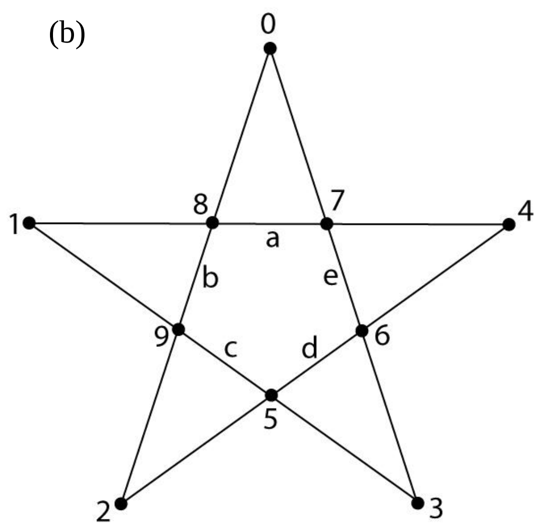


Desargues configuration
10 points, 10 lines (self-dual)

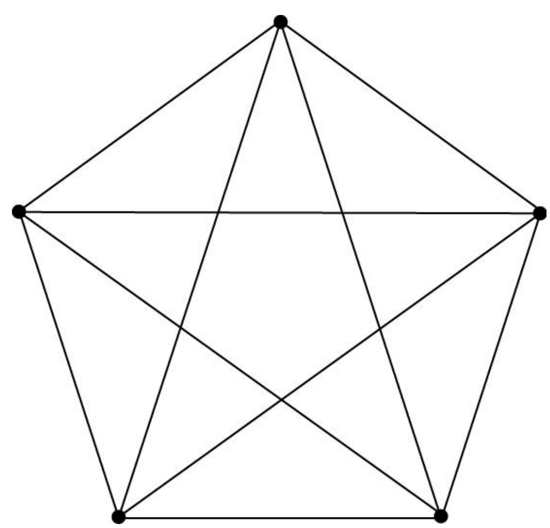


10 lines, 10 points

(b)

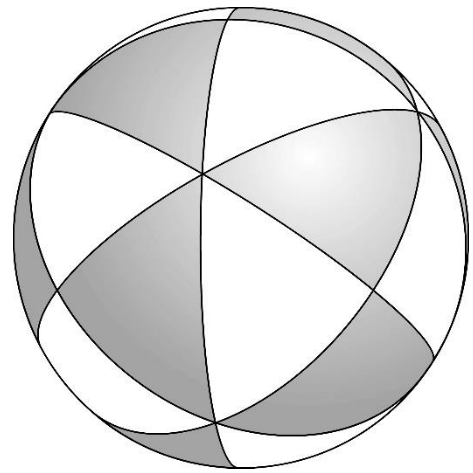


10 points, 5 lines (not self-dual)



10 lines, 5 points

4. (*10 points*) A sphere may be tiled with congruent triangles, shaded alternately white and gray as shown. Our sphere has radius $r = 1$, so its surface area is $4\pi r^2 = 4\pi$.



   a. Do triangles on the sphere have angle sum equal to, greater than, or less than π? Relate this to the curvature of the sphere (positive? negative? or zero?).

   b. Find the angle sum of each triangle (in radians).

   c. The area of a triangle on the sphere is equal to its angular excess (the amount by which the angle sum exceeds π). What is the area of each triangle?

   d. By considering areas, how many triangles are in this tiling of the sphere?

5. (*10 points*) Fill each blank below using the best single word taken from the following list. Each word from the list may be used once, more than once, or not at all.

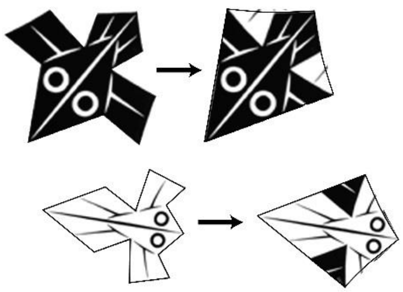
area	equal	angle	affine	addition	inversive	consistent	hyperbolic
line	point	circle	volume	physical	universal	consistency	inversion
true	field	length	similar	parallel	projective	inconsistent	intersection

Four classical types of plane geometry (affine, projective, hyperbolic and inversive) may be described either axiomatically, or constructed using coordinates taken from the ________________ of real numbers for the relevant objects (points, lines, circles, etc.). For example, the Euclidean plane is the classical __________________ plane constructed using coordinates in $\mathbb{R}$. Historically, the Euclidean plane was viewed as the single correct or true plane geometry. We now recognize, however, that no one mathematical description perfectly captures the nature of physical reality. Moreover, all these plane geometries are relatively ___________________ (in the sense that no contradiction can possibly arise from studying one of them, unless a contradiction also arises from studying the others).

For example, based on the Euclidean plane with point set $\mathbb{R}^2$, one may add a single point denoted by ∞ to obtain a representation of the _________________ plane. Likewise, within the inversive plane, starting with an arbitrary choice of ________________ γ and taking circles orthogonal to γ, one constructs a copy of the ____________________ plane. Or starting with the Euclidean plane, one may extend to the real __________________ plane by adding a set of new points, one for each ________________ class of lines; then also joining up the new points using one new _______________. If there is a contradiction to be found in any one of these four geometries, the same logical contradiction would necessarily propagate to each of the other geometries. This argument is used to show relative __________________ in the sense described above.

6. (*10 points*) The accompanying figure, *Circle Limit I*, shows a tiling of the hyperbolic plane using fish of equal area. Determine the area of each fish.

Hint: Each fish is modeled after a kite-shaped tile of the same area as shown.



Circle Limit I (M. C. Escher, 1958)

7. (*10 points*) A finite inversive plane of order n has $n^2 + 1$ points and $n^3 + n$ circles. Every circle has $n + 1$ points, and every point is on $n^2 + n$ circles. Any three distinct points lie on exactly one circle. Given any points P, Q and any circle α through P but not through Q, there exists a unique circle β passing through Q such that $\alpha \cap \beta = \{P\}$. Any two distinct circles α and β intersect in 0, 1 or 2 points; and if $|\alpha \cap \beta| = 1$, we say that the two circles are *tangent* at their point of intersection. Fill in each of the following blanks with the correct quantity (in each case a constant or a formula that depends on n).

Given any two distinct points P and Q, there are exactly ____________ circles passing through both P and Q.

If P is any point on a circle γ, then there are exactly ______________ circles tangent to γ at P.

If P is any point *not* on a circle γ, then there are exactly _______________ circles through P tangent to γ.

Given any circle γ, the number of circles α such that $|\alpha \cap \gamma| = 0$ is ________________ .

Given any circle γ, the number of circles α such that $|\alpha \cap \gamma| = 1$ is ________________ .

Given any circle γ, the number of circles α such that $|\alpha \cap \gamma| = 2$ is ________________ .

(*Hint*: We are encouraged to repeat here the counting arguments of Take-Home Test 2 which enjoyed such widespread successful performance.)

8. *(30 points)* Answer *True* or *False* to each of the following statements.

a. As predicted by general relativity (and verified experimentally), the curvature of spacetime is a variable function of the spacetime coordinates (x, y, z, t). ___________________ (*True/False*)

b. Given a set of plane tiles, there is a simple formula (based on the angles in the tiles) for determining whether or not they can tile the Euclidean plane.
___________________ (*True/False*)

c. In the real projective plane, the polarity with respect to a conic maps interior points to exterior points, and vice versa. ___________________ (*True/False*)

d. In the densest possible packing of balls of radius 1 in $\mathbb{R}^n$, each ball touches exactly $3 \times 2^{n-1}$ others. ___________________ (*True/False*)

e. Any subset of the Euclidean plane has Hausdorff dimension at most 2.
___________________ (*True/False*)

f. The Euclidean plane may be tiled with triangles having angles 30°-45°-90°.
___________________ (*True/False*)

g. A computer search has shown the nonexistence of a plane (affine, projective or inversive) of order 10. ___________________ (*True/False*)

h. The projective plane of order 2 plays a crucial role in the construction of the number system of the octonions, used in mathematical physics. ___________________ (*True/False*)

i. Given three distinct collinear points A, B, C in the real hyperbolic plane, exactly one of the points A, B, C lies between the other two. ___________________ (*True/False*)

j. In the hyperbolic plane there exists a pentagon with five 90° angles.
___________________ (*True/False*)