More on the Axioms of Plane Geometry

Consistency

Recall our axioms for affine plane geometry:

A1. Any two distinct points are on exactly one line.

A2. For any line ℓ and any point P, there is exactly one line through P not meeting ℓ.

A3. There exist four points of which no three are collinear.

In class we proved several results in affine plane geometry, including the fact that in any affine plane, any two lines have the same number of points. The number of points on any line is a constant (finite or infinite) called the *order* of the affine plane; and this number is at least 2. There exist affine planes of order 2, 3, 4, 5, 7, 8, 9, 11, 13, etc. and of infinite order. The Euclidean plane is an example of an affine plane of infinite order. There is no affine plane of order 6, a result known for many years; this can be checked by hand (but this is a lengthy exercise).


Clement Lam
1949–

In the 1980's the combined efforts of several mathematicians, primarily Clement Lam, showed that there is no affine plane of order 10. This result relies on thousands of hours of supercomputer time, adding fuel to the controversy over the legitimacy of computer proofs.

Is there an affine plane of order 12? Nobody knows.

Consider the statements

A4. There exist lines ℓ and m such that ℓ has exactly 2 points and m has exactly 3 points.

A5. There exists a line ℓ with exactly 10 points.

A6. There exists a line ℓ with exactly 12 points.

Clearly the four axioms A1, A2, A3, and A4 form an inconsistent axiomatic system since we showed (Theorem 3) that axioms A1, A2, A3 imply that any two lines have the same number of points, contrary to A4. More generally any axiomatic system from which we can possibly derive a contradiction, is inconsistent. Thus the system consisting of A1, A2, A3, and A5 is also inconsistent (if we accept the computer evidence of this) but this fact is far from obvious. Is the system consisting of axioms A1, A2, A3, and A6 consistent? Nobody knows.

Completeness and Isomorphism

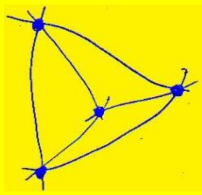
Our three axioms A1, A2 and A3 for affine plane geometry are consistent since there is at least one model for these axioms. However this system is *incomplete* because in this system there exist relevant statements which cannot be either proven or disproven. Consider for example the statement:

A7. There exists a line ℓ with exactly 2 points.

There exist affine planes in which A7 holds, and there exist affine planes in which A7 fails. Thus one can never either prove or disprove A7 using just A1, A2 and A3.

However, if we adopt A1, A2, A3 and A7 as axioms, then all models are isomorphic to the quadrangle (affine plane of order 2) we have considered. This is the affine plane which may be described in various ways, including the following:

***First Model:* A Picture**



***Second Model:* Sets**

Four points: the letters A, B, C, D.
Six lines: the subsets $\{A,B\}, \{A,C\}, \{A,D\}, \{B,C\}, \{B,D\}, \{C,D\}$.
Incidence is just ordinary set membership, e.g. the point B is on the line $\{A,B\}$ but not on the line $\{A,C\}$.

***Third Model:* Students and Classes**

Four points: the students Ann, Bob, Carl, Don.
Six lines: the classes Math, History, Geography, Biology, Music, Gym.
Here

Ann & Bob are together in Math;
Ann & Carl are in History;
Ann & Don are in Geography;
Bob & Carl are in Biology;
Bob & Don are in Music;
Carl & Don are in Gym.

Incidence refers to a student taking a class; e.g. the point Carl is not on the line Music.

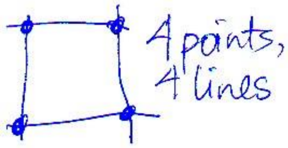
Although the interpretation of the notions of 'point', 'line' and 'incidence' differs in each case, clearly the four models are essentially equivalent, having simply different names for the various objects but the same corresponding properties. Thus any two of these models are *isomorphic*.

We may take axioms A1, A2, A3 and A7 as axioms for the affine plane of order 2. This axiomatic system is *complete* because in this system, every relevant statement can be either proven or disproven. To see which of these two cases arises, simply check whether or not the statement holds in our favorite model (any of the three models above will do). For example in the affine plane of order 2, statement A4 is clearly false as we have seen how to prove from the axioms A1, A2, A3, A7 that every line has the same number of points, in this case two.

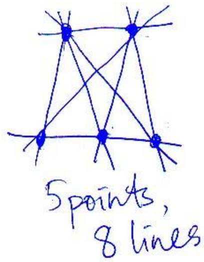
Independence

We strive for an objective understanding of our world, yet we realize that it is impossible to codify reality without at least making some assumptions. We are thereby compelled to ask: what is the smallest number of assumptions we need to make in order to generate all the explanations we could reasonably seek? (For example, what is a minimum set of assumptions about the Euclidean plane from which all other properties may be derived?) Our three axioms A1, A2 and A3 for affine plane geometry are minimal in the following sense: none of these three axioms may be derived from the other two. We know this because for each axiom there is a geometry that satisfies the other two axioms but not the one in question.

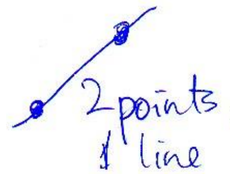
An example satisfying A2 and A3 but not A1



An example satisfying A1 and A3 but not A2



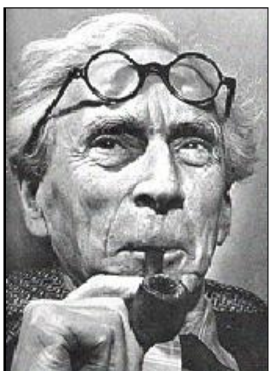
An example satisfying A1 and A2 but not A3



This shows that each of our three axioms A1, A2 and A3 is independent of the others. In any dependent axiomatic system one can always remove the redundant axioms (those that can be derived from the others) until one obtains an independent system. One possible reason for introducing redundant axioms is that by assuming these statements, we no longer have to prove them, and this will relieve some of the burden of proof at the outset in our study of such a system. But this takes us further away from our goal of appreciating what the minimal set of assumptions is. Moreover there is a danger in choosing too many axioms: this often leads to unforeseen contradictions, thereby rendering the system inconsistent.

Why do we need *axioms*?

Ideally, we would like to believe that every statement is either true or false; moreover if a statement is true, we would like to be able to prove it; if it is false, we would like to be able to disprove it. Consider however the statement: ‘This statement is false’. Is this statement true or false? If it is true, then it is false; but if it is false, then it is true. How do we avoid this paradox?



Bertrand Russell
1872–1970

In 1903 Bertrand Russell published the following set-theoretic variant of the preceding paradox: Let S be the set consisting of all sets X such that X is not an element of X. Is $S \in S$? If so, then by definition, $S \notin S$, a contradiction; on the other hand if $S \notin S$, then we must have $S \in S$, again a contradiction. In order to resolve this paradox, mathematicians realized that we needed to be more careful (less sloppy) about what we consider to be a set. A system of rules for constructing sets were proposed by Zermelo in 1908 and revised by Fraenkel and Skolem in 1922 to become what we now call the Zermelo-Fraenkel (ZF) axioms for set theory. These axioms allow us to construct all the sets we need to do modern mathematics, starting with the empty set, and then recursively forming new sets from old by well-known processes of taking subsets, power sets, Cartesian products, etc. It is hoped (and widely believed) that these axioms are consistent, i.e. that no contradiction will ever be deduced from ZF. This is important because *the entire logical foundation of modern mathematics is built on set theory*. In particular the integer and real number systems are constructed using set theory; and the Euclidean plane is constructed using real numbers (for coordinates) so it is also founded upon set theory.

Ernst Zermelo
1802–1860

Abraham Fraenkel
1891–1965

Thoraf Skolem
1777–1855

Gödel's Theorem

Unfortunately, we cannot prove that ZF is consistent (unless it is inconsistent). If ZF is inconsistent, then we can derive a contradiction from the ZF axioms; and then from a contradiction we can prove anything. (Using a contradiction we can prove every statement to be true, and we can prove every statement to be false.) So we must accept the following truth: *No one can ever prove (using the axioms for modern mathematics) that mathematics is consistent… unless it is inconsistent, in which case we can prove anything (including the consistency of modern mathematics!*). These truths are implicit in *Gödel's Incompleteness Theorem*, which states that no axiomatic system which is strong enough to include number theory can be both complete and consistent.

Kurt Gödel
1906–1978

David Hilbert
1862–1943

In the early 1900's, David Hilbert (at that time perhaps the greatest living mathematician) issued a challenge to mathematicians: Find an algorithm to decide whether a given mathematical statement was true or false, and in each case to generate a proof of the correct conclusion. However, in one of the greatest intellectual accomplishments of that century, Kurt Gödel showed that this dream was unrealizable: there exist statements in number theory (involving only integers) which can neither be proven nor disproven. Thus the axioms for modern mathematics, and in fact the ZF axioms themselves, are incomplete.

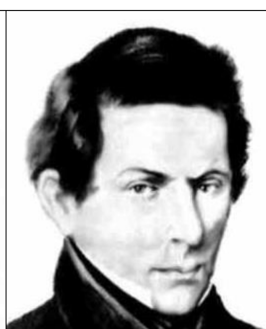
Why is this important in Geometry?

Euclid pioneered the use of the axiomatic method in his Elements, where in the third century BC he published a list of axioms ('postulates') for plane geometry. Euclid intended his axioms to be complete, consistent and independent. His fifth postulate (essentially what we have called A2) seemed so self-evident that he worried whether it might be derivable from his first four postulates. Unable to find a proof of the fifth postulate from the first four, he finally listed it among his axioms.

Euclid of Alexandria
325 BC–265 BC

Many geometers in the latter part of the second millennium tried to prove Euclid's fifth postulate from the first four. Some in fact published erroneous proofs. These were typical examples of 'circular reasoning': in their 'proofs' they assumed the very fact that they were trying to prove. Finally in the 19th century, Gauss, Lobachevsky and Bolyai found models of Euclid's first four postulates, in which the fifth postulate fails. These models describe what is now known as the *hyperbolic plane*.

János Bolyai
1802–1860



Nokolai Lobachevsky
1792–1856

Carl F. Gauss
1777–1855

In the hyperbolic plane, given a line ℓ and a point P not on ℓ, there exist many lines through P which do not meet ℓ. Some have protested that such a geometry cannot be 'true' or 'correct'. We must remember, however, that no theory is correct or incorrect; it can only be judged as consistent or inconsistent; complete or incomplete. Logically speaking, hyperbolic geometry is just as valid as Euclidean geometry; one can construct models for the hyperbolic plane using real coordinates in a manner very similar to the Cartesian description of Euclid's plane. Or one can construct models of the hyperbolic plane within Euclidean geometry, without appealing to coordinates at all. Conversely one can construct models of Euclidean geometry within hyperbolic space. This shows that Euclidean geometry is consistent if and only if hyperbolic geometry is consistent. Hyperbolic geometry has just as much right to be considered valid as does Euclidean geometry.

However, it is impossible to prove that Euclidean geometry is consistent (unless it is inconsistent). And so the same is true also for hyperbolic geometry. This is because Euclidean geometry is modeled by the real numbers, and conversely; so that Euclidean geometry is consistent if and only if real number arithmetic is consistent.

This in turn depends on the ZF axioms, which we cannot prove to be consistent.

Relative Consistency

If we have faith (and we certainly do not have proof) that the real numbers are consistent, then our model of Euclidean geometry using real coordinates will also be consistent. This is a proof of relative consistency: Euclidean geometry is just as consistent as the real number system. In the same way hyperbolic geometry is relatively consistent: we have a model of hyperbolic geometry using either Euclidean geometry or the real numbers.

But what is the 'true' plane geometry?

What does one mean by 'true'? Does one mean 'physical'? None of our ideal geometries perfectly represent physical reality, for several reasons. Ideal points and lines do not exist in nature. Moreover in the physical universe, Euclid's fifth postulate fails due to the warping of spacetime which is caused by the presence of matter. In this sense hyperbolic geometry is more accurate than Euclidean geometry as a representation of physical reality. Yet even this representation is fundamentally flawed at small scales of distance, since the real number line is totally ordered; whereas it is not physically possible to totally order positions on a physical 'line' as our idealized mental image suggests. Euclidean geometry, hyperbolic geometry, and the geometry of physical space (more correctly, physical spacetime) are all very complicated objects with surprising and subtle properties; but the subtleties of physical reality differ from the subtleties of Euclidean space.