

Affine Planes: An Introduction to Axiomatic Geometry

Here we use Euclidean plane geometry as an opportunity to introduce axiomatic systems. Keep in mind that the axiomatic approach is not the only approach to studying geometry or other mathematical subjects; and there is some argument that the value of this approach has been overrated. Nevertheless, the axiomatic method in geometry is currently a fixture, thanks to the existing curriculum standards.

Informally, a *proof* of a statement is an argument used to demonstrate the truth of that statement. We must acknowledge that every proof, however, relies on assumed notions. This applies to any axiomatic system, whether geometric, algebraic, topological, analytic, or whatever.

All that a proof can ever hope to establish is that *if* one starts from certain *assumptions*, and applies certain accepted rules of inference, *then* such-and-such follows as a logical consequence.

In studying geometry or any other mathematical subject, therefore, one must start with a choice of axioms. These may be thought of as rules of play. Just as there are many different games, each with its own rules, so there are many types of geometry, each with its own axioms. No particular choice is true, just as the rules of baseball are no more true than the rules of chess. If one wants to play chess, then in order to decide (for example) whether or not a particular move is legal, one must refer to the rules of chess; the rules of baseball are irrelevant in this case. In the same way, the statement ‘Given any line ℓ and any point P not on ℓ , there exists exactly one line through P not meeting ℓ ’ is true in Euclidean plane geometry, but false in three-dimensional Euclidean geometry.

Key ingredients in developing any subject axiomatically include:

- a) undefined notions;
- b) defined notions;
- c) axioms;
- d) statements;
- e) rules of inference;
- f) theorems;
- g) proofs; and
- h) models.

Webster’s dictionary defines ‘point’ in terms of ‘geometry’, and ‘geometry’ in terms of ‘point’. While the circular nature of this example may seem comical, it teaches us an important lesson: although it is good to define notions in terms of previously defined notions whenever possible, it is not always possible to do so; we must begin somewhere with undefined notions. Typically we will regard the notions of ‘point’ and ‘line’ as undefined, and proceed to define other notions in terms of them. For example, a *circle* (in plane geometry) may be defined as the set of points at fixed distance r from a point O ; and we then refer to r and O as the radius and center of the circle,

respectively. This definition of a circle relies on several undefined notions, including ‘point’ and ‘distance’.

To punt and admit the notions ‘point’ and ‘line’ as undefined may seem like a *weakness*; but this is not only necessary—it is what gives the axiomatic method its *strength* and *generality*. Definitions of ‘point’ and ‘line’ are not actually needed in order to prove theorems in plane geometry; all that we require is the most basic properties of points and lines, and these are listed in the axioms. And if we keep an open mind as to what the terms ‘point’ and ‘line’ might refer to (provided that they satisfy the axioms), then any theorems we prove will be portable, readily applicable in a wide variety of alternative settings, whether foreseen or not. Here is an example: when writing down the axioms for a real vector space, the terms ‘vector’ is undefined; we allow any set of vectors having well-defined vector addition and scalar multiplication that satisfies certain axioms. And so the theory applies to the set of all functions $y = f(x)$ satisfying the differential equation $\frac{d^2y}{dx^2} = -y$. Here we may say that the solution set is a 2-dimensional vector space with basis $\{\sin x, \cos x\}$, an application that would not be available to anyone for whom ‘vectors’ are strictly geometric arrows.

Statements are logical assertions. Every statement is true or false; ambiguous ‘statements’ are not permitted. An example of a statement in plane geometry is: ‘The points P , Q and R are collinear.’ This statement may be true or false, depending on the context. Some statements are quantified; for example the statement ‘For every line ℓ there exists a point not on ℓ ’ has two quantifiers: universal (‘For all ...’) and existential (‘there exists ...’). Some examples of non-statements are ‘The point P ’, ‘Every point is round’, ‘Is the point P on the line ℓ ?’, and ‘This statement is false’.

A *theorem* is a statement for which a proof is known. A *proof* of a given statement is a finite list of statements, ending with the given statement, in which every step follows either from an axiom, or a definition, or a stated hypothesis of the theorem, or a previously proved theorem, or a previous step in the proof. To decide whether a given statement does in fact follow from previous statements, one uses accepted *rules of inference*. An example of a rule of inference is ‘modus ponens’, which allows us to infer a statement B from the statements A and $A \rightarrow B$. Thus for example, if we know that ‘Alice is a teacher’ and ‘Teachers are smart’, then we may infer that ‘Alice is smart’.

A *model* is an example of something that satisfies the given axioms. A given axiomatic system may allow

- a) many different models; or
- b) no models (in which case we say the axiomatic system is *unsatisfiable* or *inconsistent*); or
- c) a unique model, up to isomorphism (in which case we say that the axiomatic system is *categorical*).

An example of (a) is the axioms of field theory, which allows for many different models (including the field \mathbb{R} of real numbers, the field \mathbb{Q} of rational numbers, the field \mathbb{C} of complex numbers, etc.). If we take the axioms of field theory and add an additional axiom stating that ‘ $1 = 2$ ’, then we obtain an unsatisfiable list of axioms, with no model. In plane geometry, there are sets of axioms that allow for more than one model; but it is possible to choose axioms in such a way that there is (up to isomorphism) exactly one model, the *Euclidean plane*.

A proper axiomatic treatment of Euclidean plane geometry is notoriously difficult and subtle. Euclid's original treatment (in *The Elements*, c. 300 BC) stood for centuries as the best attempt available; but for several reasons, it did not strictly fill the bill. In particular many of his statements were not that precise. In the early 20th century, several mathematicians, including Bertrand Russell, began an ambitious program to reformulate the foundations of modern mathematics in order to clarify the undefined and defined notions, the axioms, etc. David Hilbert and others were able to clean up Euclid's axioms considerably, while remaining faithful to the spirit of Euclid's work; and so we now have available a complete list of axioms for the Euclidean plane. However, the axiomatic description of the Euclidean plane is quite formidable, as it relies upon many undefined notions, including 'point', 'line', 'distance', 'angle', 'right angle', 'between', etc. Depending on which of the many versions of the list of axioms one refers to, this list may include such statements as

- If P , Q and R are distinct points on a line ℓ , then one (and only one) of these three points is between the other two.
- If P and Q are two distinct points on a line ℓ , then there is a unique point M on ℓ having the same distance from both P and Q .

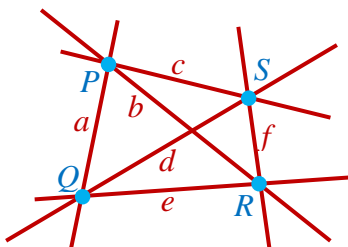
For a first experience with geometric proofs, it is advisable that one considers an axiomatic system much simpler than that required by Euclidean geometry. As an example, let us present the theory of *affine planes*.

Affine Plane Geometry

Here we take just three undefined notions: 'point', 'line', and 'incidence'. By 'incidence', we mean the relation between points and lines which, given a point P and a line ℓ , allows us to say whether or not P is *on* ℓ (or synonymously, whether or not the line ℓ *passes through* the point P); we understand that these are all just slightly different ways of saying that P is *incident with* ℓ . Note that there is no notion of distance, or of angle, or of order of points on a line, etc. In terms of these undefined notions, we may define the notion of collinearity: we say that three points P , Q and R are *collinear* if there exists a line passing through all three points P , Q and R . We take three axioms, as follows:

- A1. Given any two distinct points P and Q , there is a unique line passing through both P and Q .
- A2. Given any line ℓ and any point P not on ℓ , there exists exactly one line through P not meeting ℓ .
- A3. There exist four points of which no three are collinear.

An affine plane is any structure of points and lines with incidence satisfying these axioms. The Euclidean plane satisfies these axioms, i.e. it is a model; but there are many other models as well, the smallest of which is the affine plane of order 2 which includes just four **points** and six **lines**:



Don't take this picture too seriously; it is merely a pictorial way of saying that there are just four points P, Q, R, S and six lines a, b, c, d, e, f ; and that incidence is given by: P lies on a, b and c only; Q lies on a, d and e only; etc. There is no point of intersection of lines b and d ; this picture may be somewhat misleading. This example is known as an *affine plane of order 2* or a *quadrangle*; later we will define the order of an affine plane. An easy application of the axioms shows that every affine plane contains such a quadrangle (i.e. affine subplane of order 2).

Before proving theorems about affine planes, let's make a useful definition: Two lines are said to be *parallel* if either they do not meet (i.e. they have no points in common), or they are the same line. We write $\ell \parallel m$ if the lines ℓ and m are parallel.

Another useful definition: given distinct points P and Q , denote by PQ the unique line through P and Q (see A1). If ℓ and m are non-parallel lines, then we denote by $\ell \cap m$ the unique point on both ℓ and m (such a point exists since ℓ and m are not parallel; and it is unique by A1).

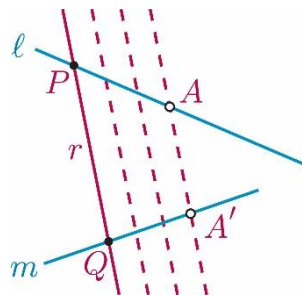
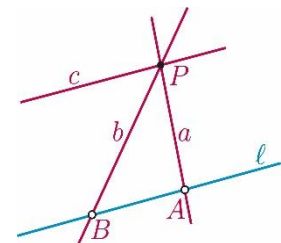
Theorem 1. In any affine plane, given lines r, s and t with $r \parallel s$ and $s \parallel t$, then $r \parallel t$. Thus parallelism is an equivalence relation on the set of lines.

Proof. We may assume that the lines r, s and t satisfy $r \parallel s$ and $s \parallel t$. Without loss of generality, we may suppose that r and t are *not* parallel, in which case there exists a point P on both r and t . But then the uniqueness of the line through P parallel to s (see axiom A2) forces $r = t$, so that the desired conclusion $r \parallel t$ follows. To see that this means that parallelism of lines is an equivalence relation, recall that by definition, every line is parallel to itself by definition; also if $\ell \parallel \ell'$ then $\ell' \parallel \ell$. □

Theorem 2. In any affine plane, every line has at least two points; and any two lines have the same number of points.

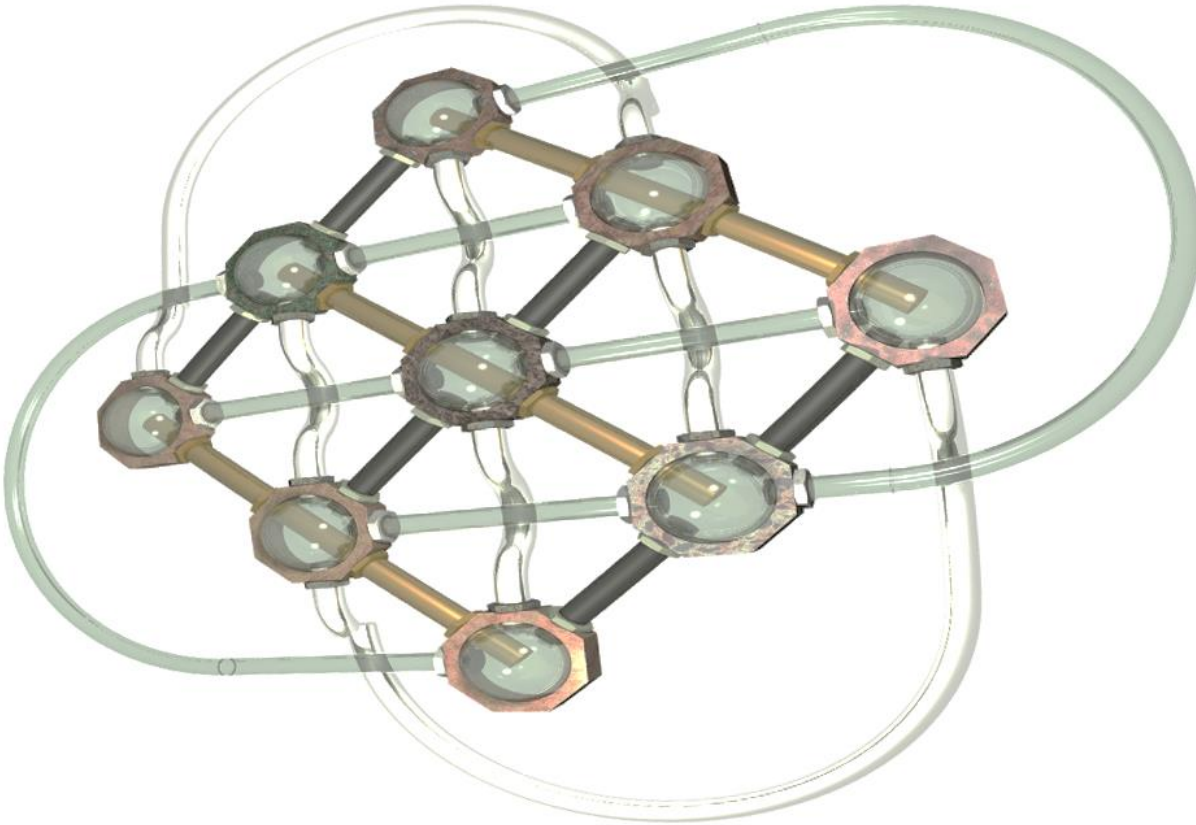
(We mean that if one line has n points, then every line has n points; if one line has infinitely many points, then every line has infinitely many points. More precisely, given any two lines ℓ and m , the points on ℓ correspond bijectively with the points on m .)

Proof. Let ℓ be any line in a given affine plane. By the observation above (the existence of an affine subplane of order 2), there exists a point P not on ℓ , and at least three lines (call them a, b, c as in the picture) through P . By A2, at most one of the lines a, b, c is parallel to ℓ , so we may say a and b are not parallel to ℓ . Now $A := a \cap \ell$ and $B := b \cap \ell$ are two points of ℓ . Moreover $A \neq B$ since $PA = a \neq b = PB$. This proves the first assertion.



Let ℓ and m be distinct lines. By the first assertion, there exists a point P on ℓ but not on m ; and a point Q on m but not on ℓ . Let $r = PQ$. For every point A on ℓ , there is a unique line through A parallel to r , and this line meets m in a unique point A' . We have a bijective correspondence $A \leftrightarrow A'$ between the points of ℓ and the points of m . (It is irrelevant whether or not ℓ is parallel to m ; the same correspondence works in either case.) □

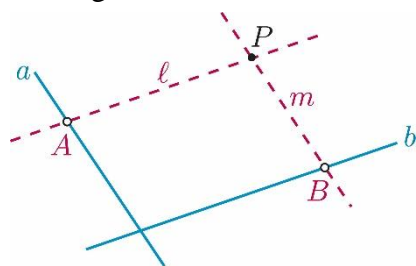
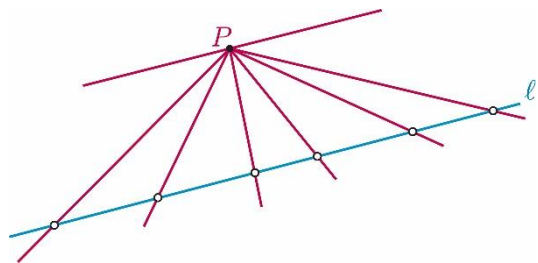
The number of points on any line of an affine plane (which, by Theorem 2, is a constant) is called the *order* of the plane. We have seen an affine plane of order 2; and below we show an affine plane of order 3. It has 9 points and 12 lines, with 3 points on each line and 4 lines through each point.



We have seen the cases $n = 2, 3$ as special cases of the following result.

Theorem 3. In any affine plane of order n , there are n^2 points and $n(n + 1)$ lines. Every line has n points and every point is on n lines. Every parallel class consists of n lines.

Proof. Consider an affine plane of order n , so that every line has exactly n points, as we have seen. Let P be any point, and let ℓ be any line not through P . (Such a line ℓ certainly exists; for example every quadrangle contains such a line.) There is exactly one line through P parallel to ℓ ; and each of the remaining lines through P meets ℓ in a unique point. So the number of lines through P must be $n + 1$, one more than the number of points on ℓ .



Now choose two intersecting lines a and b . (Two intersecting lines can, for example, be chosen from a given quadrangle.) There is a bijective correspondence between points P in the plane and pairs of lines (ℓ, m) where $\ell \parallel a$ and $m \parallel b$, given by $P = \ell \cap m$. Since there are n choices of ℓ (one through each point A of a) and n choices of m (one through each point B of b),

this makes n^2 choices of point P in the plane. (Note that we allow A , or B , or both, to coincide with the point $\ell \cap m$.)

Since each parallel class of lines partitions all the n^2 points of the plane into lines each of size n , there must be exactly n lines in each parallel class.

Next we count, in two different ways, the number of incident point-line pairs (P, ℓ) . Since there are n^2 choices of point P and $n + 1$ lines through each, there must be exactly $n^2(n + 1)$ such pairs (P, ℓ) . On the other hand, if \mathcal{L} is the set of lines then there are $|\mathcal{L}|$ choices of ℓ , and n points on each line, so altogether $n|\mathcal{L}|$ pairs (P, ℓ) . Solving $n|\mathcal{L}| = n^2(n + 1)$ yields $|\mathcal{L}| = n(n + 1)$ for the number of lines. \square

The Classical Affine Planes

The most obvious affine planes which we call the *classical affine planes* are those coordinatized by fields. A *field* is a number system F having addition, subtraction, multiplication, and division satisfying all the usual properties of commutativity, associativity and distributivity. For example, we have the field \mathbb{R} of real numbers; the field \mathbb{Q} of rational numbers; the field \mathbb{C} of complex numbers; and for each prime p , we have the field $\mathbb{F}_p = \{0, 1, 2, \dots, p - 1\}$ of integers mod p . If F is any field (such as \mathbb{R} , \mathbb{Q} , \mathbb{C} , or \mathbb{F}_p) then we construct the *affine plane* $AG_2(F)$ over F ('affine geometry of dimension 2 over F ') as follows:

- Points are ordered pairs $(x, y) \in F^2$.
- Lines are defined as follows:
 - 'Non-vertical' lines, which have the form $\{(x, y) : y = mx + b\}$ where $m, b \in F$.
 - 'Vertical' lines, which have the form $\{(a, y) : y \in F\}$ where $a \in F$.
- Incidence is the usual set membership.

The real affine plane $AG_2(\mathbb{R})$ is the usual Euclidean plane. This coordinate construction does a very good job of describing what one needs to know about the Euclidean plane; and it answers a wide range of the questions we might ask about plane geometry. However, we want to understand the Euclidean plane not just algebraically and analytically, but in many other ways—synthetic, axiomatic, etc.

In the finite case, namely given a field $F = \mathbb{F}_q$ with q elements (here $q = p^r$ where p is prime and r is a positive integer), the affine plane $AG_2(\mathbb{F}_q)$ has q^2 points (x, y) (there are q choices for x , and q choices for y); similarly, there are $q(q + 1)$ lines (q vertical lines and q non-vertical lines; q points on each line; etc. Note that the order of the plane $AG_2(\mathbb{F}_q)$ is q , the same as the order $|\mathbb{F}_q| = q$ of the field coordinatizing it.

There are many affine planes other than the classical ones. For example, there are seven affine planes of order 9, up to isomorphism, including the classical one coordinatized by the field \mathbb{F}_9 of order 9.