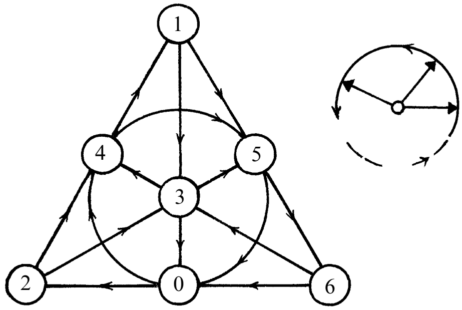

Fig. 16.3 The Fano plane $\mathbb{P}^2(\mathbb{F}_2)$, with 7 points and 7 lines (the circle counting as a 'straight line') numbered according to Fig. 16.1a. This provides the multiplication table for the basis elements $\mathbf{i}_0$, $\mathbf{i}_1$, $\mathbf{i}_2, \ldots, \mathbf{i}_6$ of the octonion division algebra, where the arrows provide the cyclic ordering that gives a '+' sign.

its scope as a geometry is rather limited, it plays an important role of a different kind, in providing the multiplication law for *octonions* (see §11.2, §15.4). The Fano plane has 7 points in it, and each point is to be associated with one of the generating elements $\mathbf{i}_0$, $\mathbf{i}_1$, $\mathbf{i}_2$, ..., $\mathbf{i}_6$ of the octonion algebra. Each of these is to satisfy $\mathbf{i}_r^2 = -1$. To find the product of two *distinct* generating elements, we just find the line in the Fano plane which joins the points representing them, and then the remaining point on the line is the point representing the product (up to a sign) of these other two. For this, the simple picture of the Fano plane is not quite enough, because the *sign* of the product needs to be determined also. We can find this sign by reverting to the description given by the disc, depicted in Fig. 16.1a, or by using the (equivalent) arrow arrangements (intrepreted cyclicly) of Fig. 16.3. Let us assign a cyclic ordering to the marked points on the disc—say anticlockwise. Then we have $\mathbf{i}_x\mathbf{i}_y = \mathbf{i}_z$ if the cyclic ordering of $\mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ agrees with that assigned by the disc, and $\mathbf{i}_x\mathbf{i}_y = -\mathbf{i}_z$ otherwise. In particular, we have $\mathbf{i}_0\mathbf{i}_1 = \mathbf{i}_3 = -\mathbf{i}_1\mathbf{i}_0$, $\mathbf{i}_0\mathbf{i}_2 = \mathbf{i}_6$, $\mathbf{i}_1\mathbf{i}_6 = -\mathbf{i}_5$, $\mathbf{i}_4\mathbf{i}_2 = -\mathbf{i}_1$, etc.[16.6]

Although there is a considerable elegance to these geometric and algebraic structures, there seems to be little obvious contact with the workings of the physical world. Perhaps this should not surprise us, if we adopt the point of view expressed in Fig. 1.3, in §1.4. For the mathematics that has any direct relevance to the physical laws that govern our universe is but a tiny part of the Platonic mathematical world as a whole—or so it would seem, as far as our present understanding has taken us. It is possible that,

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[16.6] Show that the 'associator' $a(bc) - (ab)c$ is antisymmetrical in a, b, c when these are generating elements, and deduce that this (whence also $a(ab) = a^2b$) holds for *all* elements. *Hint*: Make use of Fig. 16.3 and the full symmetry of the Fano plane.