Test

Monday, November 4, 2024

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time allowed: 50 minutes. Total value: 100 points (plus 37 bonus points).

1. Let $\alpha, \beta, \gamma \in \mathbb{C}$ be the three roots of the polynomial $f(x) = x^3 - 7x + 2 \in \mathbb{Q}[x]$.

   (a) *(10 points)* Express

   $$\tfrac{1}{\alpha} = \square + \square\,\alpha + \square\,\alpha^2 \in \mathbb{Q}[\alpha]$$

   by finding the three missing coefficients in $\mathbb{Q}$.

   (b) *(15 points)* Evaluate (as rational numbers in simplified form)

   $$\alpha + \beta + \gamma = \square$$

   $$\alpha\beta\gamma = \square$$

   $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \square$$

2. *(32 points)* For each of the following, give an *explicit example*:

(a) An extension field $F \supset \mathbb{Q}$ of degree $[F : \mathbb{Q}] = 3$ having only one automorphism (the identity or trivial automorphism).

(b) An extension field $E \supset \mathbb{Q}$ of degree $[E : \mathbb{Q}] = 3$ having more than one automorphism.

(c) An irreducible polynomial $f(x) \in \mathbb{Q}[x]$ having a rational root.

(d) A reducible polynomial $g(x) \in \mathbb{Q}[x]$ having no rational roots.

(e) A subring $R \subset \mathbb{Q}^{2\times 2}$ isomorphic to the field $\mathbb{Q}[\sqrt{5}]$. (Here $\mathbb{Q}^{2\times 2}$ denotes the ring of all 2×2 matrices with entries in $\mathbb{Q}$.)

(f) A basis of the field extension $\mathbb{Q}[\alpha] \supset \mathbb{Q}$ where α is a root of the irreducible polynomial $m(x) = x^3 - 7x + 2$.

(g) Two subfields $F_1, F_2 \subset \mathbb{C}$ such that $F_1 \cong F_2$, with $F_1 \subset \mathbb{R}$ but $F_2 \not\subset \mathbb{R}$.

(h) An extension $E \supset \mathbb{Q}$ (of degree at least two) and two elements $\alpha \neq \beta$ in E, such that $\mathbb{Q}[\alpha] = \mathbb{Q}[\beta] = E$.

3. Let $f(x) = x^4 - 8x^2 + 11$. You may use the fact that $f(x)$ is irreducible in $\mathbb{Q}[x]$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be its four roots in $\mathbb{C}$.

   (a) *(10 points)* One of the four roots of $f(x)$ is $\alpha = \sqrt{4 + \sqrt{5}}$. So we might as well assume $\alpha = \alpha_1$. Find the other three roots $\alpha_2, \alpha_3, \alpha_4$ and write them as explicitly as we have written the first root.

Let $E = \mathbb{Q}[\alpha]$. Note that each of the four roots of $f(x)$ is algebraic of degree 4 over $\mathbb{Q}$, so that $\mathbb{Q}[\alpha_j] = E$ for all $j \in \{1, 2, 3, 4\}$, and $[E : \mathbb{Q}] = 4$.

   (b) *(10 points)* Is $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ a basis for E over $\mathbb{Q}$? Explain.

(c) *(12 points)* Find three positive integers a, b, c such that $\{1, \sqrt{a}, \sqrt{b}, \sqrt{c}\}$ is a basis for E over $\mathbb{Q}$.

Since E is the splitting field of $f(x)$ over $\mathbb{Q}$, the extension $E \supset \mathbb{Q}$ is Galois and its group $G = \mathsf{Aut}\, E$ of automorphisms has order $|G| = [E : \mathbb{Q}] = 4$.

(d) *(8 points)* Is the group G cyclic (of order 4) or a Klein four-group? Give a *very short* explanation by appealing to the evident similarity with other examples we have done in class.

(e) *(10 points)* Let $\beta = \sqrt{2+i} + \sqrt{2-i}$ where $i = \sqrt{-1}$. Show that $\sqrt{2+i} \notin E$; and that $\beta \in E$.

4. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) Every subfield of $\mathbb{C}$ is infinite. ______*(True/False)*

(b) The extension field $\mathbb{Q}[2^{1/3}] \supset \mathbb{Q}$ contains $2^{1/2}$. ______*(True/False)*

(c) If $E \supset \mathbb{Q}$ is a finite extension field with $1 < [E : \mathbb{Q}] < \infty$, then there are infinitely many intermediate fields K satisfying $E \supset K \supset \mathbb{Q}$. ______*(True/False)*

(d) Every proper subfield $F \subset \mathbb{R}$ (i.e. a subfield which is not all of $\mathbb{R}$) is an extension of finite degree $[F : \mathbb{Q}] < \infty$. ______*(True/False)*

(e) Every element of $\mathbb{C}$ has a square root in $\mathbb{C}$. ______*(True/False)*

(f) If σ is an automorphism of an extension field $F \supseteq \mathbb{Q}$, then $\{a \in F : \sigma(a) = a\}$ is a subfield of F. ______*(True/False)*

(g) If $F \supset \mathbb{Q}$ is an extension field of degree $n = [F : \mathbb{Q}] < \infty$, and $\alpha \in F$, then α is a root of a monic polynomial $f(x) \in \mathbb{Q}$ of degree at most n. ______*(True/False)*

(h) The ring of all 2×2 real matrices is a field. ______*(True/False)*

(i) If σ is an automorphism of a field extension $F \supseteq \mathbb{Q}[\sqrt{2}]$, then $\sigma(\sqrt{2}) \in \{\sqrt{2}, -\sqrt{2}\}$. ______*(True/False)*

(j) For every positive integer n, there exists an irreducible polynomial $f(x) \in \mathbb{R}[x]$ of degree n. ______*(True/False)*