University of Wyoming
Math 4520—Fall 2024
Algebra III
Fields
Department of Mathematics

Solutions to the Test
November, 2024

1. (a) Divide the equation $\alpha^3-7\alpha+2=0$ by α and solve to obtain $\frac{1}{\alpha}=\frac{7}{2}+0\alpha+(-\frac{1}{2})\alpha^2$.
   (b) Expand $x^3-7x+2=(x-\alpha)(x-\beta)(x-\gamma)$ and equate coefficients to obtain
   $$\begin{aligned}\alpha+\beta+\gamma &= 0,\\ \alpha\beta+\alpha\gamma+\beta\gamma &= -7,\\ \alpha\beta\gamma &= -2,\\ \tfrac{1}{\alpha}+\tfrac{1}{\beta}+\tfrac{1}{\gamma} &= \tfrac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma} = \tfrac{-7}{-2} = \tfrac{7}{2}.\end{aligned}$$
   Compare: #6,7 on Practice Problems 1.

2. Know your examples.
   (a) $F=\mathbb{Q}[\alpha]$ where $\alpha=2^{1/3}$ (example done in class).
   (b) $E=\mathbb{Q}[\alpha]$ where $\alpha^3+\alpha^2-2\alpha-1=0$ (example done in class). A second example (from practice problems) instead takes $\alpha^3-3\alpha+1=0$.
   (c) $x-7$
   (d) $x^4+x^2+1=(x^2+x+1)(x^2-x+1$. Or $x^4+2x^2+1=(x^2+1)^2$.
   (e) $R=\{aI+bA\,:\,a,b\in\mathbb{Q}\}$ where $I=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, $A=\left[\begin{smallmatrix}0&5\\1&0\end{smallmatrix}\right]$
   (f) $\{1,\alpha,\alpha^2\}$ is the standard basis
   (g) In our notation for the splitting field of x^3-2, we used $\alpha=2^{1/3}$ and $\omega=e^{\frac{2\pi i}{3}}=\frac{1+\sqrt{-3}}{2}$. Take $F_1=\mathbb{Q}[\alpha]$ and $F_2=\mathbb{Q}[\alpha\omega]$.
   (h) An easy example is $\alpha=\sqrt{2}$ and $\beta=1+\sqrt{2}$.

3. (a) $\alpha_2=-\sqrt{4+\sqrt{5}}$, $\alpha_3=\sqrt{4-\sqrt{5}}$, $\alpha_4=-\sqrt{4-\sqrt{5}}$.
   (b) No; they are not linearly independent since $\alpha_1+\alpha_2=0$ and $\alpha_3+\alpha_4=0$.
   (c) Note that $\sqrt{5}=\alpha_1^2-5\in E$ and $\sqrt{11}=\alpha_1\alpha_3\in E$. Multiplying these also gives $\sqrt{55}\in E$. At this point, we see that $\{1,\sqrt{5},\sqrt{11},\sqrt{55}\}$ is a basis for E, just as in our example of $\mathbb{Q}[\sqrt{2},\sqrt{5}]$ done in class.
   (d) G is a Klein four-group (compare with $Q[\sqrt{2},\sqrt{5}]$ from class).
   (e) Of course $\beta\notin E$ since $E\subset\mathbb{R}$ whereas β is not real. But $\beta^2=(2+i)+(2-i)+2\sqrt{(2+i)(2-i)}=4+2\sqrt{5}\in E$ since $\sqrt{5}\in E$. Unfortunately I made a typo on this question so we don't actually get $\beta\in E$, sorry.

4. (a) T (b) F (c) F (d) F (e) T (f) T (g) T (h) F (i) T (j) F

Some comments and explanations, provided for your benefit only (not required for answering #4):

(a) Every subfield of $\mathbb{C}$ contains $\mathbb{Q}$.

(b) Since $[\mathbb{Q}[2^{1/3}] : \mathbb{Q}] = 3$ does not divide $[\mathbb{Q}[2^{1/2}] : \mathbb{Q}] = 2$, $\mathbb{Q}[2^{1/2}]$ cannot be a subfield of $\mathbb{Q}[2^{1/3}]$ (by the transitivity of degrees of field extensions).

(c) Every number field (finite extension of $\mathbb{Q}$) has only finitely many subfields. As discussed in class, this follows from the fact that a vector space over an infinite field cannot be the union of finitely many proper subspaces.

(d) The extension $\mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots] \supset \mathbb{Q}$ is an extension of infinite degree. So is the subfield of $\mathbb{R}$ consisting of all real algebraic numbers.

(e) Given $\alpha \in \mathbb{C}$, the polynomial $x^2 - \alpha \in \mathbb{C}[x]$ has a root by the Fundamental Theorem of Algebra.

(f) This was observed during our discussion of Galois theory. But in fact it is elementary to prove that $K = \{a \in F : \sigma(a) = a\}$ is a subfield of F: show that it contains 0 and 1; and whenever $a, b \in K$, it is easy to see that $a, b, a \pm b, ab \in K$ (same for $\frac{a}{b} \in K$, if $b \neq 0$).

(g) The $n+1$ elements $1, \alpha, \alpha^2, \ldots, \alpha^n \in F$ must be linearly dependent over $\mathbb{Q}$, so there exist $c_0, c_1, \ldots, c_n \in \mathbb{Q}$, not all zero, such that $a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0$. We may suppose that the last nonzero coefficient in this list of coefficients is a_k; then dividing by a_k, we obtain a monic polynomial of degree $k \in \{1, 2, \ldots, n\}$ with rational coefficients, having α as a root.

(h) The ring of 2×2 real matrices is not commutative, and it has zero divisors.

(i) As explained in class, σ must permute the roots of $x^2 - 2 \in \mathbb{Q}[x]$.

(j) Every irreducible polynomial in $\mathbb{R}[x]$ has degree 1 or 2. This is a corollary of the Fundamental Theorem of Algebra; see the handout on Complex Numbers for details.