Math 4520

Fields

Book 1

A **field** is a number system with $+, -, \times, \div, 0, 1$ in which we can divide by every nonzero element.

$\mathbb{R} = \{\text{real numbers}\}$ is a field

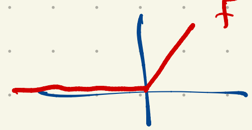
$\mathbb{C} = \{\text{complex numbers}\}$ $\quad$ " " $\qquad$ $\mathbb{C} = \{a+bi : a,b \in \mathbb{R}\}$, $\quad i = \sqrt{-1}$

$\mathbb{Q} = \{\text{rational numbers}\}$ $\quad$ " " $\qquad$ $\mathbb{Q} = \{\frac{a}{b} : a,b \in \mathbb{Z},\ b \neq 0\}$

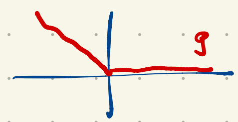
$\mathbb{Z} = \{\text{integers}\} = \{\ldots,-3,-2,-1,0,1,2,3,4,\ldots\}$ is not a field. $\quad$ $\mathbb{Z}$ is a commutative ring with 1 (having no zero divisors)

Other examples of fields?

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$



$$g(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



$f(x) + g(x) = |x|$

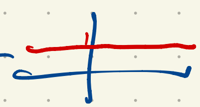
$f(x) - g(x) = x$

$f(x)g(x) = \underline{\underline{0}}$

$\cancel{\dfrac{f(x)}{g(x)}}$

$f(x)$ and $g(x)$ are nonzero functions whose product is 0 (these are called <u>zero divisors</u> in ring theory).

The continuous functions $\mathbb{R} \to \mathbb{R}$ form a commutative ring with identity but not a field.



$\mathbb{F}_5 = \{0,1,2,3,4\}$ is a field of <u>order</u> 5.

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$3+4=2$ $\qquad$ $3\times 4 = 2$

$3-4=4$ $\qquad$ $\frac{3}{4} = 2$

$\mathbb{R}, \mathbb{C}, \mathbb{Q}$ are infinite fields; $\mathbb{F}_5$ is a finite field.

In $\mathbb{Z}$, $8 \equiv 3 \bmod 5$. (8 is congruent to 3 mod 5)

In $\mathbb{F}_5$, $8 = 3$.

$$\frac{3}{4} = 2 \iff 3 = 2\times 4$$

$2\times 4 = 8 = 3$