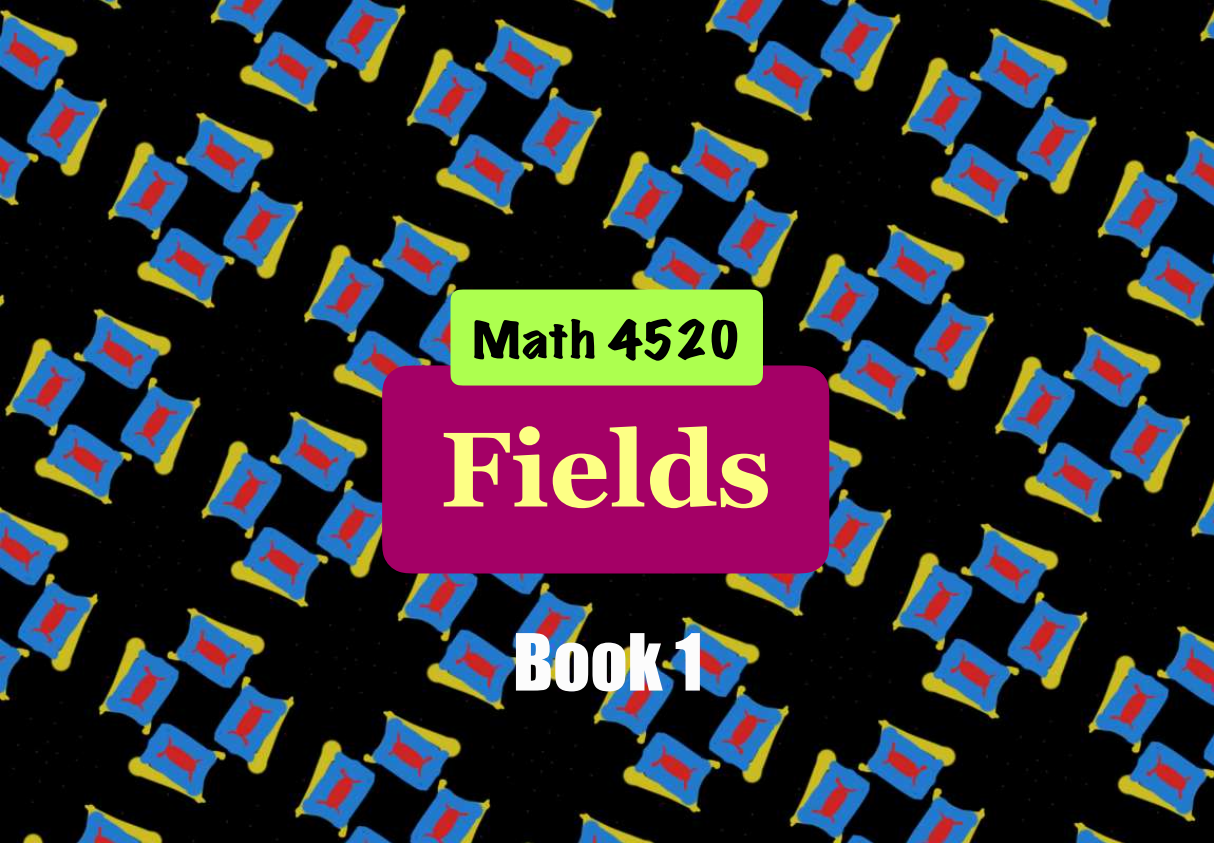




Math 4520

Fields

Book 1

A field is a number system with $+$, $-$, $\times$, $\div$, 0 , 1 in which we can divide by every nonzero element.

$\mathbb{R} = \{\text{real numbers}\}$ is a field

$\mathbb{C} = \{\text{complex numbers}\}$ " " "

$\mathbb{Q} = \{\text{rational numbers}\}$ " " "

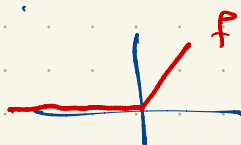
$\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$, $i = \sqrt{-1}$

$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$

$\mathbb{Z} = \{\text{integers}\} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$ is not a field. $\mathbb{Z}$ is a commutative ring with 1 (having no zero divisors)

Other examples of fields?

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$



$$g(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



$$f(x) + g(x) = |x|$$

$$f(x) - g(x) = x$$

$$f(x)g(x) = 0$$

$$f \neq 0$$

$f(x)$ and $g(x)$ are nonzero functions whose product is 0 (these are called zero divisors in ring theory).

The continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ form a commutative ring with identity  but not a field.

$\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ is a field of order 5.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$$3 + 4 = 2$$

$$3 - 4 = 4$$

$$3 \times 1 = 2$$

$$\frac{3}{4} = 2$$

$\mathbb{R}, \mathbb{C}, \mathbb{Q}$ are infinite fields; $\mathbb{F}_5$ is a finite field.
In $\mathbb{Z}$, $8 \equiv 3 \pmod{5}$. (8 is congruent to 3 mod 5)
In $\mathbb{F}_5$, $8 = 3$.

$$\frac{3}{4} = 2$$



$$3 = 2 \times 4$$

$$2 \times 4 = 8 = 3$$

Fields

Let F be a set containing distinct elements called 0 and 1 (thus $0 \neq 1$). Suppose addition, subtraction, multiplication and division are defined for all elements of F (except division by 0 is not defined).

Thus $a + b, a - b, ab, \frac{a}{d} \in F$ whenever $a, b, d \in F$ and $d \neq 0$.

Define $-a = 0 - a$.

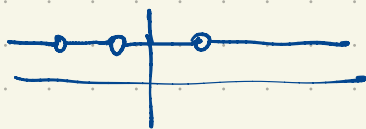
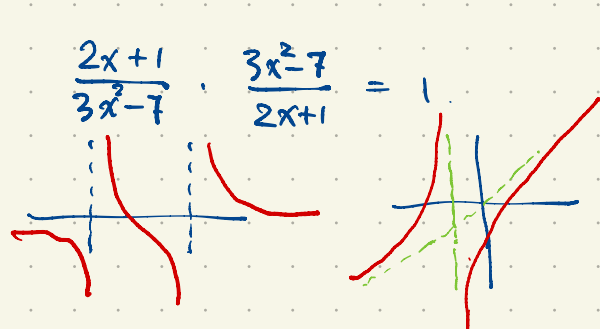
If the following properties are satisfied by *all* elements $a, b, c, d \in F$ with $d \neq 0$, then F is a **field**.

$a + b = b + a$	$a + (b + c) = (a + b) + c$	$ab = ba$
$a + 0 = a$	$a(bc) = (ab)c$	$1a = a$
$a + (-a) = 0$	$a(b + c) = ab + ac$	$\frac{a}{d} d = a$
$a + (-b) = a - b$		

The ring of polynomials in x with real coefficients is

$\mathbb{R}[x]$ is a commutative ring with identity having no zero divisors (if $f(x), g(x)$ are nonzero polynomials then $f(x)g(x)$ is a nonzero polynomial)

$\mathbb{R}(x) = \{ \text{rational functions in } x \text{ with real coefficients} \}$, a field. $\frac{1}{f(x)} = f(x)^{-1}$



$$\mathbb{R}(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{R}[x], g(x) \neq 0 \right\}$$

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} : a, b \in \mathbb{Q} \}$ is a field.

$$\alpha = 3 + 7\sqrt{2}, \beta = 5 - \sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

$$\alpha + \beta = (3 + 7\sqrt{2}) + (5 - \sqrt{2}) = 8 + 6\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

$$\alpha - \beta = (3 + 7\sqrt{2}) - (5 - \sqrt{2}) = -2 + 8\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

$$\alpha\beta = (3 + 7\sqrt{2})(5 - \sqrt{2}) = 15 - 3\sqrt{2} + 35\sqrt{2} - 14 = 1 + 32\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

$$\frac{\alpha}{\beta} = \frac{3 + 7\sqrt{2}}{5 - \sqrt{2}} \cdot \frac{5 + \sqrt{2}}{5 + \sqrt{2}} = \frac{15 + 35\sqrt{2} + 35\sqrt{2} + 14}{25 - 2} = \frac{29 + 70\sqrt{2}}{23} = \frac{29}{23} + \frac{70}{23}\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

$$\mathbb{Q}[i] = \{ a + bi : a, b \in \mathbb{Q} \}, i = \sqrt{-1}$$

This is a field.

$$\alpha = 1 + \frac{1}{3}i, \beta = \frac{3}{2} - i \Rightarrow \alpha + \beta = \frac{5}{2} - \frac{2}{3}i$$

$$\alpha - \beta = -\frac{1}{2} + \frac{4}{3}i$$

$$\alpha\beta = \frac{3}{2} - i + \frac{1}{2}i + \frac{1}{3} = \frac{11}{6} - \frac{1}{2}i \in \mathbb{Q}[i]$$

$$\frac{\alpha}{\beta} = \frac{1 + \frac{1}{3}i}{\frac{3}{2} - i} \cdot \frac{6}{6} = \frac{6 + 2i}{9 - 6i} \cdot \frac{9 + 6i}{9 + 6i} = \frac{54 + 36i + 18i - 12}{81 + 36} = \frac{42 + 54i}{117} = \frac{14}{39} + \frac{18}{39}i \in \mathbb{Q}[i]$$

Example $R = \{ \text{real } 2 \times 2 \text{ matrices} \} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ noncommutative ring with identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ having zero divisors

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The subset $F = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{R} \right\} \subset R$ is a subring which is actually a field

$$\det \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a^2 + b^2 \neq 0 \text{ whenever } \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$F = \{ \underbrace{aI + bJ} : a, b \in \mathbb{R} \}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ basis for } F.$$

$$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$J^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$(aI + bJ)(cI + dJ) = acI + adJ + bcJ + bdJ^2 = (ac - bd)I + (ad + bc)J \in F$$

$$(cI + dJ)(aI + bJ) = (ca - db)I + (cb + da)J$$

All field axioms hold in F , so F is a field.

Rename element $aI + bJ \in F$ as $aI + bi = a + bi \in \mathbb{C}$
 $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad a, b \in \mathbb{R}$

$$\text{In } \mathbb{C}: (a + bi)(c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i$$

Our example F is essentially the same as $\mathbb{C}$. More precisely, they are isomorphic: we have a bijection $f: \mathbb{C} \rightarrow F$, $f(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for all $a, b \in \mathbb{R}$ such that

$$f(zw) = f(z)f(w), \quad f(z + w) = f(z) + f(w) \quad \text{for all } z, w \in \mathbb{C}.$$

$\mathbb{Q}[i]$ is isomorphic to the ring $E = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b, c, d \in \mathbb{Q} \right\}$

$$f: \mathbb{Q}[i] \rightarrow E, \quad f(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$\alpha = 1 + \frac{1}{3}i$ $\{1, i\}$ basis for $\mathbb{Q}[i]$ over $\mathbb{Q}$

$$\beta = \frac{3}{2} - i$$

$$f(\beta) = \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & \frac{3}{2} \end{bmatrix}$$

$$f(\beta)^{-1} = \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & \frac{3}{2} \end{bmatrix}^{-1} = \frac{1}{\frac{9}{4} + 1} \begin{bmatrix} \frac{3}{2} & 1 \\ -1 & \frac{3}{2} \end{bmatrix} = \frac{4}{13} \begin{bmatrix} \frac{3}{2} & 1 \\ -1 & \frac{3}{2} \end{bmatrix}$$

$$f\left(\frac{\alpha}{\beta}\right) = f(\alpha) f(\beta)^{-1} = \frac{4}{13} \begin{bmatrix} 1 & \frac{1}{3} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 1 \\ -1 & \frac{3}{2} \end{bmatrix} = \frac{4}{13} \begin{bmatrix} \frac{7}{6} & \frac{3}{2} \\ -\frac{3}{2} & \frac{7}{6} \end{bmatrix} = \frac{4}{13} \cdot \frac{1}{6} \begin{bmatrix} 7 & 9 \\ -9 & 7 \end{bmatrix} = \frac{2}{39} \begin{bmatrix} 7 & 9 \\ -9 & 7 \end{bmatrix} = \frac{1}{39} \begin{bmatrix} 14 & 18 \\ -18 & 14 \end{bmatrix}$$

$$\frac{3}{2} - \frac{1}{3} = \frac{9-2}{6} = \frac{7}{6}$$

$$f\left(\frac{14}{39} + \frac{18}{39}i\right) = \frac{1}{39} \begin{bmatrix} 14 & 18 \\ -18 & 14 \end{bmatrix} \quad \checkmark$$

$\mathbb{Q}[\sqrt{2}] \cong K = \left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} : a, b \in \mathbb{Q} \right\}$ basis for K : $I, J = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $J^2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$

$$\det \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} = a^2 - 2b^2 \neq 0 \quad \text{if } a, b \in \mathbb{Q} \text{ are not both zero.}$$

$$\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}^{-1} = \frac{1}{a^2 - 2b^2} \begin{bmatrix} a & -b \\ -2b & a \end{bmatrix}$$

Note: $\left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a commutative ring with identity having zero divisors

$$\begin{bmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 1 \\ 2 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ 2b & a \end{bmatrix} = aI + bJ, \quad J = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad J^2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$$

$\alpha = \sqrt[3]{2} = 2^{1/3}$ $1, \alpha, \alpha^2$ is a basis: every element of $\mathbb{Q}[\alpha]$ is uniquely expressible as a linear combination $a \cdot 1 + b \cdot \alpha + c \cdot \alpha^2$ $a, b, c \in \mathbb{Q}$

$\mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$ (field)

Compare $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ (field) $a, b, c \in \mathbb{Q}$

$$\alpha^3 = 2$$

$$\begin{aligned}\alpha^3 &= 2 \\ \alpha^4 &= 2\alpha \\ \alpha^5 &= 2\alpha^2 \\ \alpha^6 &= 4\end{aligned}$$

$\mathbb{Q}[\alpha]$ is a 3-dimensional vector space over the scalars $a, b, c \in \mathbb{Q}$

If $E \supseteq F$ is a field extension (F is a subfield of E)

$$\begin{aligned}u &= 2 + \alpha + \alpha^2 \\ v &= 1 - \alpha + \alpha^2\end{aligned}$$

$$u + v = 3 + 2\alpha^2$$

$$u - v = 1 + 2\alpha$$

$$uv = (2 + \alpha + \alpha^2)(1 - \alpha + \alpha^2) = 2 - \alpha + 2\alpha^2 + \alpha^4 = 2 - \alpha + 2\alpha^2 + 2\alpha = 2 + \alpha + 2\alpha^2$$

$$\frac{u}{v} = \frac{2 + \alpha + \alpha^2}{1 - \alpha + \alpha^2} = \underbrace{a + b\alpha + c\alpha^2}_{\text{rational numbers}} \Rightarrow 2 + \alpha + \alpha^2 = (1 - \alpha + \alpha^2)(a + b\alpha + c\alpha^2)$$

$$\begin{aligned}&= a + (b - a)\alpha + (a - b + c)\alpha^2 + (b - c)\alpha^3 + c\alpha^4 \\&= a + (b - a)\alpha + (a - b + c)\alpha^2 + 2(b - c) + 2c\alpha \\&= (a + 2b - 2c) + (-a + b + 2c)\alpha + (a - b + c)\alpha^2\end{aligned}$$

$$a + 2b - 2c = 2$$

$$-a + b + 2c = 1$$

$$a - b + c = 1$$

$$\begin{aligned}\left[\begin{array}{ccc|c}1 & 2 & -2 & 2 \\ -1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 1\end{array}\right] &\sim \left[\begin{array}{ccc|c}1 & 2 & -2 & 2 \\ 0 & 3 & 0 & 3 \\ 1 & -1 & 1 & 1\end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c}1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & 3 & -1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & 3 & -1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 2\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3\end{array}\right] \xrightarrow{\begin{matrix} \leftarrow a \\ \leftarrow b \\ \leftarrow c \end{matrix}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3\end{array}\right]\end{aligned}$$

$$\frac{u}{v} = \frac{4}{3} + \alpha + \frac{2}{3}\alpha^2 \quad \checkmark$$

$$3.65152175(2)$$

Check: $\alpha \approx 1.25992104989$

$u \approx 4.84732210186$

$v \approx 1.32748000207$

$\frac{u}{v} \approx 3.6515217512$

A good way to check!

$$\mathbb{Q}[\alpha]_{\alpha=2^{1/3}} \cong \{aI + bA + cA^2 : a, b, c \in \mathbb{Q}\} = \left\{ \begin{bmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{bmatrix} : a, b, c \in \mathbb{Q} \right\} = \mathbb{Q}[A]$$

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I$$

$$A^4 = 2A$$

An isomorphism $f: \mathbb{Q}[\alpha] \rightarrow \mathbb{Q}[A]$

is given by $f(a + b\alpha + c\alpha^2) = aI + bA + cA^2$

$$\text{eg, } f(u) = f(2 + \alpha + \alpha^2) = 2I + A + A^2 = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$f(v) = f(1 - \alpha + \alpha^2) = I - A + A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$f(uv) = f(u)f(v) = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} = 2 + A + 2A^2$$

$$f\left(\frac{u}{v}\right) = f(u)f(v)^{-1} = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}^{-1}$$

$\mathbb{C} \supset \mathbb{R}$ is a field extension

$\mathbb{C} = \mathbb{R}[i] = \{a + bi : a, b \in \mathbb{R}\}$ has basis $\{1, i\}$ over $\mathbb{R}$

$\mathbb{R} \supset \mathbb{Q}$ $1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{10}, \dots$

$\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$

Consider the prime field $\mathbb{F}_{101} = \{0, 1, 2, \dots, 100\}$ with $\alpha = 89$, $\beta = 23$. (Addition, multiplication, etc. mod 101)

$$\alpha + \beta = 112 = 11$$

$$\alpha - \beta = 66$$

$$\alpha\beta = 2047 = 27$$

$$\frac{\beta}{\alpha} = 23 \cdot 42 = 966 = 57$$

$$23 \cdot 42 \times \alpha = \beta \quad ?$$

$$42 \cdot \alpha = 1 \quad ?$$

101 is prime

$$\frac{\beta}{\alpha} = \beta \cdot \frac{1}{\alpha} = 23 \cdot 42 = 966 = 57$$

How do we compute inverses in $\mathbb{F}_p = \{0, 1, 2, \dots, p-1\}$, p prime?
Review: basic properties of $\mathbb{Z}$ (handout on website)

$$\gcd(89, 101) = 1$$

89 has (positive) divisors $(1, 89)$
101 has (positive) divisors $(1, 101)$

the only common number is 1
(common divisor of 89 and 101)

$$\begin{array}{cc} 3 \cdot 5 & 3 \cdot 7 \\ \downarrow & \downarrow \\ \gcd(15, 21) & = 3 \end{array}$$

Divisors of 15: $-15, -5, -3, -1, 1, 3, 5, 15$

Divisors of 21: $-21, -7, -3, -1, 1, 3, 7, 21$

Common divisors of 15 and 21: $-3, -1, 1, 3$

greatest common divisor of 15 and 21: 3.

If $b = ra$ ($a, b, r \in \mathbb{Z}$) then we say

- a divides b
- a is a divisor of b
- b is a multiple of a
- a is a factor of b
- $a \mid b$

vertical bar

$$3 \mid 15 \quad 7 \nmid 15 \quad 7 \text{ does not divide } 15$$

$$3 < 5 \quad 8 \nmid 5$$

$$0 \mid 0$$

$$0 = -17 \cdot 0$$

Divisors of 1: $-1, 1$

Divisors of 0: $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$

$3 \mid 0$ because $0 = 0 \cdot 3$

Fact: Given $a, b \in \mathbb{Z}$, not both zero, $\gcd(a, b) = r \cdot a + s \cdot b$ for some $r, s \in \mathbb{Z}$.
 We can "easily" compute $\gcd(a, b)$, r, s , using the extended Euclidean algorithm.

eg. $\gcd(89, 101) = \square = \boxed{42} \cdot 89 + \boxed{-37} \cdot 101$.

$$101 = 1 \cdot 89 + 12$$

$$89 = 7 \cdot 12 + 5$$

$$12 = 2 \cdot 5 + 2$$

$$5 = 2 \cdot 2 + 1$$

$$2 = 2 \cdot 1 + 0$$

$$\leftarrow \gcd(101, 89) = 1$$

$$1 = 5 - 2 \cdot 2$$

$$= 5 - 2 \cdot (12 - 2 \cdot 5)$$

$$= 5 \cdot 5 - 2 \cdot 12$$

$$= 5 \cdot (89 - 7 \cdot 12) - 2 \cdot 12$$

$$= 5 \cdot 89 - 37 \cdot 12$$

$$= 5 \cdot 89 - 37 \cdot (101 - 89)$$

$$= \boxed{42} \cdot 89 - \boxed{37} \cdot 101$$

check: $3738 - 3737 = 1$

$$42 \cdot 89 \equiv 1 \pmod{101} \quad \text{in } \mathbb{Z}$$

$$42 \cdot 89 = 1 \quad \text{in } \mathbb{F}_{101}$$

$$\begin{aligned} 21 &= 1 \cdot 15 + 6 \\ 15 &= 2 \cdot 6 + \boxed{3} \leftarrow \gcd(21, 15) = 3 \\ 6 &= 2 \cdot 3 + 0 \end{aligned}$$

$$3 = \boxed{42} \cdot 21 + \boxed{3} \cdot 15 \quad ?$$

$$\begin{aligned} 3 &= 15 - 2 \cdot 6 \\ &= 15 - 2 \cdot (21 - 15) \\ &= 3 \cdot 15 - 2 \cdot 21 \end{aligned}$$

Check: $45 - 42 = 3$.

It is faster to keep track of the coefficients only:

101	89	
1	0	101
0	1	89
1	-1	12
-7	8	5
15	-17	2
-37	42	$\boxed{1}$
*	*	0

$$\begin{aligned} 3x + 5y &= 13 \\ 7x - 11y &= 29 \end{aligned}$$

x	y	
3	5	13
7	-11	29

The last non-zero remainder is the gcd

$$\leftarrow \gcd(101, 89) = 1 = -37 \cdot 101 + 42 \cdot 89$$

Euclid's Algorithm (for finding the gcd of two numbers) relies on repeated application of the Division Algorithm.

Theorem: Let $n, d \in \mathbb{Z}$ with $d > 0$. Then $n = q \cdot d + r$ for some $q, r \in \mathbb{Z}$ which are unique.
 $0 \leq r < d$

$\mathbb{R}[x]$ = ring of all polynomials in x with real coefficients

$\mathbb{R}[x]^* = \{\text{units of } \mathbb{R}[x]\} = \{\text{nonzero constant polynomials}\}$

A ^{nonzero} element $f \in \mathbb{R}[x]$ (or any commutative ring with identity) is reducible if $f = gh$ for some $g, h \in \mathbb{R}[x]$, where neither g nor h is a unit.

$$x^2 + 1 = gh \quad ?$$

$$x^2 + 1 = (x^2 + 1) \cdot 1$$

$$x^2 + 1 = (5x^2 + 5) \cdot \left(\frac{1}{5}\right) \quad \text{unit}$$

$$x^2 + 1 = gh \Rightarrow g \text{ or } h \text{ is a unit. Why?}$$

$(x^2 + 1 \text{ is irreducible})$

If $f(x) \in \mathbb{R}[x]$ then the degree of $f(x)$ is the largest n such that the coefficient of x^n in $f(x)$ is nonzero: (denoted $\deg f(x)$)

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n, \quad a_n \neq 0 \quad (a_0, a_1, \dots, a_n \in \mathbb{R})$$

$$\deg(x^2 + 1) = 2$$

$$\deg(g(x)h(x)) = \deg g(x) + \deg h(x)$$

$$\begin{aligned} \deg(x) &= 1 \\ \deg(5) &= 0 \\ \deg(0) & \end{aligned}$$

$$0 \cdot (x^2 + 1) = 0$$

$$\deg(0) + \deg(x^2 + 1) = \deg 0 \quad \text{This is a problem.}$$

One approach says: $\deg f(x)$ is only defined for nonzero polynomials.

The other approach: $\deg 0 = -\infty$

If $x^2 + 1 = g(x)h(x)$, $g(x), h(x) \in \mathbb{R}[x]$ then the factors $g(x), h(x)$ have degrees 0, 2 or 1, 1 or 2, 0.

But degree 0 polynomials are nonzero constants (i.e. units).

Otherwise $\deg g(x) = \deg h(x) = 1$ i.e. $g(x) = ax + b$, $h(x) = cx + d$, $a, b, c, d \in \mathbb{R}$, $a \neq 0$, $c \neq 0$.

Both $\mathbb{Z}$ and $\mathbb{R}[x]$ are commutative rings with identity having no zero divisors. Neither is a field.
 Units = {invertible elements} = $\{\pm 1\}$ or {nonzero real constants} respectively.

Both rings have the Division Algorithm:

In $\mathbb{Z}$: For all $n, d \in \mathbb{Z}$ with $d > 0$, there exist unique $q, r \in \mathbb{Z}$ such that $n = \underset{\substack{\uparrow \\ \text{divisor}}}{q}d + \underset{\substack{\uparrow \\ \text{remainder}}}{r}$, $0 \leq r < d$.

In $\mathbb{R}[x]$: For all $f(x), d(x) \in \mathbb{R}[x]$ with $d(x) \neq 0$, there exist unique $q(x), r(x) \in \mathbb{R}[x]$ such that

$$f(x) = q(x)d(x) + r(x), \quad \deg r(x) < \deg d(x).$$

Euclid's Algorithm works in both $\mathbb{Z}, \mathbb{R}[x]$. (It's just more ugly in $\mathbb{R}[x]$.)

$2x+1 = 2(x+\frac{1}{2})$
 irreducible

In $\mathbb{Z}$, $\gcd(a, b) = ra + sb$ for some $r, s \in \mathbb{Z}$. Here $a, b \in \mathbb{Z}$, not both zero.

In $\mathbb{R}[x]$, $\gcd(f(x), g(x))$ is the largest monic polynomial divisor of both $f(x), g(x)$.
 $\uparrow$ the leading term has coefficient 1.

$$\gcd(\underset{\substack{\uparrow \\ (3x+1)x}}{3x^2+x}, \underset{\substack{\uparrow \\ 4 \cdot x}}{4x}) = x.$$

$$\gcd(\underset{\substack{\uparrow \\ (x+\frac{1}{3}) \cdot 3}}{3x+1}, \underset{\substack{\uparrow \\ (x+\frac{1}{3}) \cdot 9x}}{9x^2+3x}) = x + \frac{1}{3}$$

We want to require $\gcd(f(x), g(x))$ to be the monic polynomial of largest degree dividing both $f(x)$ and $g(x)$.

$$\gcd(f(x), g(x)) = r(x)f(x) + s(x)g(x) \quad \text{for some } r(x), s(x) \in \mathbb{R}[x].$$

If F is any field then $F[x] = \{\text{polynomials in } x \text{ with coefficients in } F\}$ works the same way.

Euclidean rings: eg. $\mathbb{Z}, \mathbb{R}[x], F[x], \dots$ (integral domain which has the Division Algorithm).

Let $f(x) \in F[x]$ where F is a field. ($f(x)$ is a polynomial in x with coefficients in F).

A root of $f(x)$ is a value a such that $f(a) = 0$. (Often $a \in F$ or $a \in$ some larger field.)

eg. $x^2 + 1$ has roots $\pm i$ in $\mathbb{C}$

If $f(x) \in F[x]$ has a root $a \in F$ then $x-a$ divides $f(x)$, i.e. $f(x) = (x-a)q(x)$ for some $q(x) \in F[x]$.

Roots of $f(x) \longleftrightarrow$ factors of $f(x)$ of degree 1 (i.e. linear factors)

polynomials of degree 0 are constant
 1 linear
 2 quadratic
 3 cubic
 4 quartic

In one direction: if $x-a \mid f(x)$ ($x-a$ divides $f(x)$) then $f(a) = (a-a)q(a) = 0$
 so a is a root in F .

Conversely, if $a \in F$ is a root of $f(x)$, we get a linear factor $x-a$ because using the Division Algorithm,

$$f(x) = q(x)(\underbrace{x-a}_{\deg(x-a)=1}) + r(x), \quad q(x), r(x) \in F[x], \quad \deg r(x) < 1 \quad \text{so } \deg r(x) \leq 0$$

i.e. $r(x) = r \in F$ (a constant)

$$f(x) = q(x)(x-a) + \underbrace{r}_{\text{constant}} \quad 0 = f(a) = \underbrace{q(a)}_0 (a-a) + r \Rightarrow r=0. \Rightarrow f(x) = q(x)(x-a).$$

$\Rightarrow x-a \mid f(x)$.

Factor $f(x) = x^3 - x^2 + 2x - 2 = (x-1)(x^2+2)$ factored into irreducible factors in $\mathbb{Z}[x]$, $\mathbb{Q}[x]$, $\mathbb{R}[x]$

$f(1) = 1 - 1 + 2 - 2 = 0$

$f(x) = (x-1)(x+i\sqrt{2})(x-i\sqrt{2})$ in $\mathbb{C}[x]$
 Linear factors

If $f(x)$ has degree 1, 2 or 3 then every factorization of $f(x)$ has a linear factor.

If $\deg f(x) = 4$ then it is possible that $f(x)$ might factor into irreducible ^{quadratic} factors with no roots in F .

Eg. in $\mathbb{R}[x]$, $x^4 + 1$ has no real roots. Is it irreducible in $\mathbb{R}[x]$?

$$x^4 + 1 = (x^2 + ax + c)(x^2 - ax + c) \quad \text{Finish on/before Monday.}$$

$$c = \pm 1, \quad 2c - a^2 = 0 \quad (\text{coefficient of } x^2), \quad a^2 = 2, \quad a = \pm\sqrt{2} \\ c = 1$$

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1) \quad (\text{irreducible factors } x^2 \pm \sqrt{2}x + 1) \quad \text{discriminant } (\sqrt{2})^2 - 4 \cdot 1 \cdot 1 = -2$$

What are the irreducible polynomials in $\mathbb{R}[x]$?

$$x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}) \quad \text{reducible in } \mathbb{R}[x]$$

The irreducible polynomials in $\mathbb{R}[x]$ are

$$\text{discriminant } 0^2 - 4(1)(-2) = 8 > 0$$

- all polynomials of degree 1, i.e. $ax + b$, $a, b \in \mathbb{R}$, $a \neq 0$
- all polynomials of the form $ax^2 + bx + c$, $a, b, c \in \mathbb{R}$, $a \neq 0$, $\underbrace{b^2 - 4ac}_{\text{discriminant}} < 0$

$x^2 - 2$ is irreducible in $\mathbb{Q}[x]$ but reducible in $\mathbb{R}[x]$

$$x^2 - 2x + 1 = (x - 1)^2 \quad \text{reducible} \\ (-2)^2 - 4 \cdot 1 \cdot 1 = 0$$

Polynomials in $\mathbb{R}[x]$ of degree 3 are reducible since using Calculus I or using a little algebra (roots of polynomials in $\mathbb{R}[x]$ come in pairs under complex conjugation).

Polynomials in $\mathbb{R}[x]$ of degree 4 are always reducible by the fact that $\mathbb{C}$ is algebraically closed:

Every polynomial in $\mathbb{C}[x]$ of degree ≥ 1 has a root in $\mathbb{C}$. (Fundamental Theorem of Algebra).

Irreducible polynomials in $\mathbb{C}[x]$ are the polynomials of degree 1.

In $\mathbb{Q}[x]$ there are irreducible polynomials of every degree $n \geq 1$, eg. $x^n - 2$ is irreducible for every $n \geq 1$.

Eg. $x^4 - 2 \in \mathbb{Q}[x]$ is irreducible.

If $x^4 - 2 = g(x)h(x)$ where $g(x), h(x) \in \mathbb{Q}[x]$ but neither $g(x)$ nor $h(x)$ is a unit. Get contradiction on Wednesday...

Let's try $m(x) = x^3 + 2x + 2 \in \mathbb{Z}[x]$. Show: this is irreducible in $\mathbb{Z}[x]$.

Proof by contradiction: Any factorization of $m(x)$ in $\mathbb{Z}[x]$ with non-constant factors has the form

$$x^3 + 2x + 2 = (x + a)(x^2 + bx + c), \quad ac = 2. \Rightarrow (a, c) = (\pm 1, \pm 2) \text{ or } (\pm 2, \pm 1)$$

$$\text{or } (-x)(-x^2) \quad (\text{we can ignore this case since it simplifies to } (x \dots)(x^2 \dots))$$

Since $m(1) = 5$, $m(-1) = -1$, $m(2) = 4$, $m(-2) = -10$ are all nonzero, this is a contradiction.

General fact: If $m(x) \in \mathbb{Z}[x]$ is monic irreducible in $\mathbb{Z}[x]$, then $m(x)$ is also irreducible in $\mathbb{Q}[x]$.
This is an elementary (but slightly tricky) theorem in ring theory. We will use this fact freely.
So: $m(x) = x^3 + 2x + 2$ is irreducible in $\mathbb{Q}[x]$.

Eg. Show $m(x) = x^4 + 2x + 2 \in \mathbb{Q}[x]$ is irreducible. It suffices to prove irreducibility in $\mathbb{Z}[x]$.

Prove this by contradiction. If $m(x)$ factors nontrivially in $\mathbb{Z}[x]$, then either

Case (i) $x^4 + 2x + 2 = (x + a)(x^3 + bx^2 + cx + d)$, $a, b, c, d \in \mathbb{Z}$, or

Case (ii) $x^4 + 2x + 2 = (x^2 + ax + b)(x^2 + cx + d)$, $a, b, c, d \in \mathbb{Z}$.

Case (i) leads to an easy contradiction: $ad = 2 \Rightarrow a \in \{\pm 1, \pm 2\}$ and $m(-a) = 0$.

Here $m(1) = 5$, $m(-1) = 1$, $m(2) = 22$, $m(-2) = 14$ are nonzero, a contradiction.

In Case (ii), $a + c = 0$ so $x^4 + 2x + 2 = (x^2 + ax + b)(x^2 - ax + d)$,

$$bd = 2 \Rightarrow (b, d) = (1, 2), (2, 1), (-1, -2) \text{ or } (-2, -1)$$

$$b + d - a^2 = 0$$

$$ad - ab = 2$$

$$a(d - b) = 2$$

$$bd = 2$$

If $b = 1$ and $d = 2$ then $a = 2$ and

$$b + d - a^2 = 1 + 2 - 2^2 = -1, \text{ a contradiction.}$$

If $b = 2$ and $d = 1$ then $a = -2$ and $b + d - a^2 = 2 + 1 - (-2)^2 = -1$, again a contradiction.

These cases cannot hold since $b + d = a^2 \geq 0$.

A very standard construction of a field extension $E \supseteq F$, starting from a known field F , is to start with $m(x) \in F[x]$ irreducible of degree n . Declare θ to be a root of $m(x)$ and $E = F[\theta] = \{a_0 + a_1\theta + a_2\theta^2 + \dots + a_{n-1}\theta^{n-1} : a_0, \dots, a_{n-1} \in F\}$. Then E is a field. It is an extension of degree n with basis $1, \theta, \theta^2, \dots, \theta^{n-1}$.

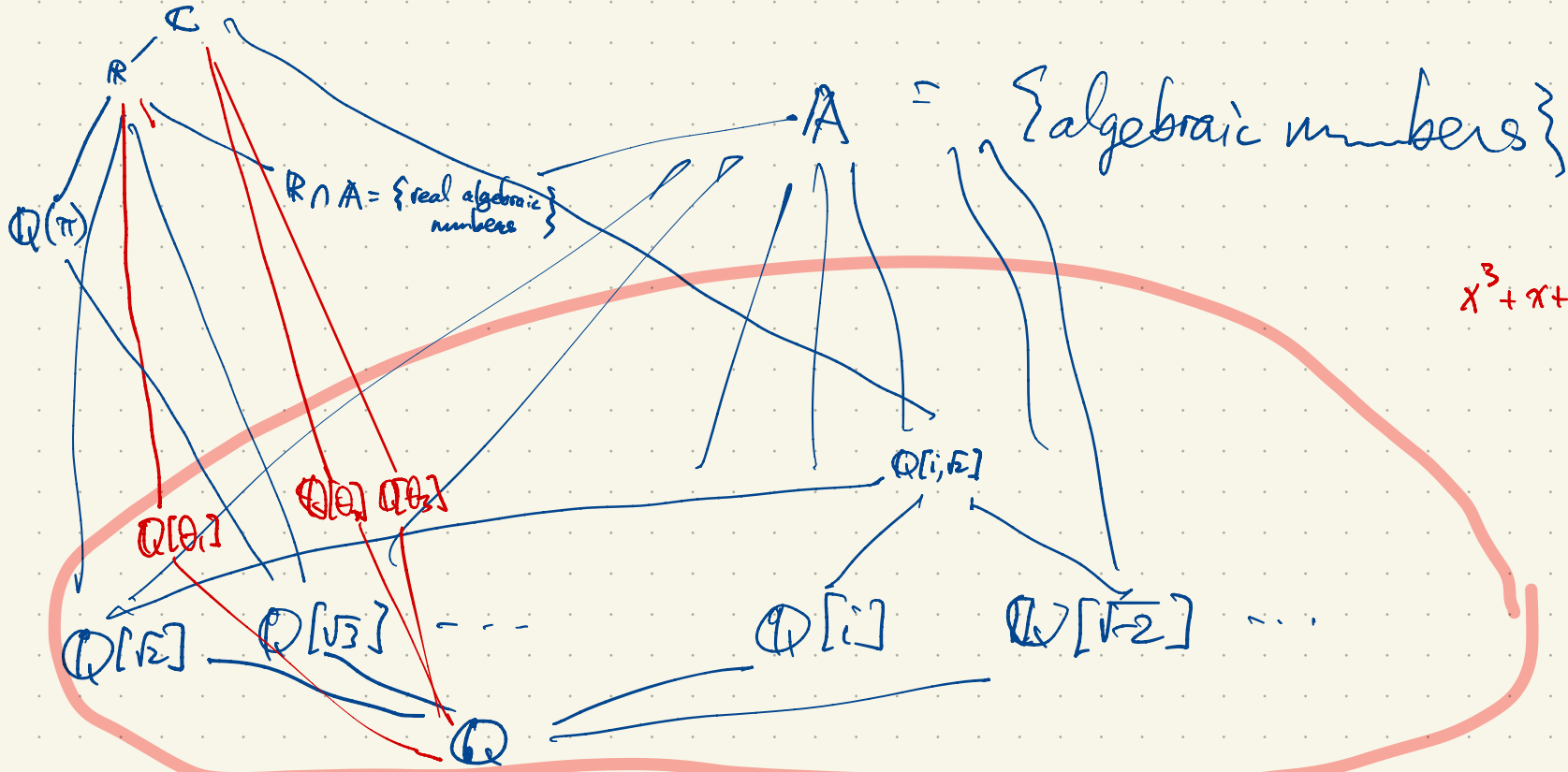
If $m(x) = g(x)h(x)$, $g(x), h(x) \in F[x]$ of degree $< n$, then $m(\theta) = 0 = \underbrace{g(\theta)h(\theta)}_{\text{nonzero elements of } E}$

Every finite degree $E \geq \mathbb{R}$ field extension is either $E = \mathbb{R}$ or $E = \mathbb{C}$.
(dimension as a vector space)

there are extensions $E \supset \mathbb{R}$ of infinite degree eg. $\mathbb{R}(x) = \{ \text{rational functions with real coefficients} \}$.

The only finite extension of $\mathbb{C}$ is $\mathbb{C}$. $\mathbb{C}(x) \supset \mathbb{C}$ is an infinite extension (degree/dimension is infinite).

There are finite extension fields $E \supset \mathbb{Q}$ of every degree. These examples are called (algebraic) number fields.



$$x^3 + x + 3 = (x - \theta_1)(x - \theta_2)(x - \theta_3)$$

$\theta = \theta_1$
 $\theta_3 = \overline{\theta_2}$

An element $x \in \mathbb{C}$ is algebraic if it is a root of some nonzero poly. $f(x) \in \mathbb{Q}[x]$.

If x is not algebraic, then x is transcendental. eg. $\pi, e, \dots$

Most real and complex numbers are transcendental because $|\mathbb{Q}| = |\mathbb{A}| =$ countably infinite but $|\mathbb{R}| = |\mathbb{C}|$ is uncountable.

Cardinality proves the existence of transcendental numbers.

How can we prove a number is transcendental?

$\sqrt{2}$ is irrational but algebraic, not transcendental.

Prove that $\sqrt{2}$ is irrational: Suppose $\sqrt{2} = \frac{a}{b}$ as a reduced fraction, a, b positive integers, $\gcd(a, b) = 1$.

Then $2 = \frac{a^2}{b^2}$ so $a^2 = 2b^2$ is even, so a is even: $a = 2c$, $c \in \mathbb{Z}$. so $4c^2 = 2b^2$ i.e. $2c^2 = b^2$ is even.

so $b = 2d$, $d \in \mathbb{Z}$. Then $\gcd(a, b) \geq 2$, a contradiction.

This proof that $\sqrt{2} \notin \mathbb{Q}$ is due to Euclid.

Proof that e is irrational: Suppose $e = \frac{a}{b}$, a, b positive integers, $\gcd(a, b) = 1$. Then

$$a = be = b \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots \right) = b \sum_{k=0}^{\infty} \frac{1}{k!} = b \left(1 + 1 + \frac{1}{2} + \dots + \frac{1}{b!} + \frac{1}{(b+1)!} + \frac{1}{(b+2)!} + \dots \right)$$

$$0! = 1$$

$$1! = 1$$

$$2! = 1 \times 2 = 2$$

$$3! = 1 \times 2 \times 3 = 6$$

$$4! = 1 \times 2 \times 3 \times 4 = 24$$

etc.

$$(b-1)! a = b \left((b-1)! + (b-1)! + \frac{(b-1)!}{2!} + \frac{(b-1)!}{3!} + \dots + \frac{(b-1)!}{(b-1)!} + \frac{(b-1)!}{b!} + \frac{(b-1)!}{(b+1)!} + \dots \right)$$

integers

$$\text{integers} = \text{integers} + 1 + \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \dots$$

integer between 0 and 1, a contradiction.

In HW1, θ is a root of $m(x) = x^3 + x + 3$ i.e. $\theta^3 + \theta + 3 = 0 \Rightarrow \theta^3 = -3 - \theta$

$$F = \mathbb{Q}[\theta] = \{a + b\theta + c\theta^2 : a, b, c \in \mathbb{Q}\}$$

$$(\theta+1)^{-1} = \frac{1}{\theta+1} = a + b\theta + c\theta^2$$

$$1 = (a + b\theta + c\theta^2)(1 + \theta)$$

$$= a + (a+b)\theta + (b+c)\theta^2 + c\theta^3$$

$$= a + (a+b)\theta + (b+c)\theta^2 + c(-3-\theta)$$

$$= \underbrace{(a-3c)}_1 + \underbrace{(a+b-c)}_0 \theta + \underbrace{(b+c)}_0 \theta^2$$

$$\begin{aligned} a - 3c &= 1 \\ a + b - c &= 0 \\ b + c &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$$\Rightarrow a = -2, b = 1, c = -1 \Rightarrow (\theta+1)^{-1} = -2 + \theta - \theta^2$$

$$\text{Check } (\theta+1)(-2+\theta-\theta^2) = -2-\theta-\theta^3 = -2-\theta-(-3-\theta) = 1 \quad \checkmark$$

Similar to #4: let's find a ^{monic} cubic polynomial in $\mathbb{Q}[x]$ of degree ≤ 3 having α as a root.

i.e. $g(x) = x^3 + ax^2 + bx + c$, $a, b, c \in \mathbb{Q}$ such that $g(\alpha) = 0$ where $\alpha = 1 + \theta + \theta^2$.

(The answer is unique.)

$$\alpha = 1 + \theta + \theta^2$$

$$\begin{aligned} \alpha^2 &= (1 + \theta + \theta^2)^2 = 1 + \theta^2 + \theta^4 + 2\theta + 2\theta^2 + 2\theta^3 = 1 + \theta^2 + (-3\theta - \theta^2) + 2\theta + 2\theta^2 + 2(-3-\theta) \\ &= -5 - 3\theta + 2\theta^2 \end{aligned}$$

$$\begin{aligned} \alpha^3 &= \alpha^2 \cdot \alpha = (1 + \theta + \theta^2)(-5 - 3\theta + 2\theta^2) = -5 - 8\theta - 6\theta^2 - \theta^3 + 2\theta^4 \\ &= -5 - 8\theta - 6\theta^2 - (-3-\theta) + 2(-3\theta - \theta^2) \\ &= -2 - 13\theta - 12\theta^2 \end{aligned}$$

$$\text{We need } g(\alpha) = 0 = \alpha^3 + a\alpha^2 + b\alpha + c \Rightarrow a\alpha^2 + b\alpha + c = -\alpha^3$$

$$\Rightarrow a(-5 - 3\theta + 2\theta^2) + b(1 + \theta + \theta^2) + c = -2 - 13\theta - 12\theta^2$$

$$\Rightarrow \underbrace{(-5a + b + c)}_{-2} + \underbrace{(-3a + b)}_{-13} \theta + \underbrace{(2a + b)}_{-12} \theta^2 = -2 - 13\theta - 12\theta^2$$

The smallest subfield of $\mathbb{R}$ containing π is $\mathbb{Q}(\pi)$.

Any subfield $E \subseteq \mathbb{C}$ contains $\mathbb{Q}$.

Assuming a subfield $E \subseteq \mathbb{C}$ contains π then it must contain $1, \pi, \pi^2, \pi^3, \pi^4, \dots$ so E contains every expression of the form

$$a_0 + a_1\pi + a_2\pi^2 + a_3\pi^3 + \dots + a_n\pi^n, \quad n \geq 0, \quad a_0, a_1, \dots, a_n \in \mathbb{Q}.$$

ie. $E \supseteq \mathbb{Q}[\pi] = \{ \text{all polynomials in } \pi \text{ with rational coefficients} \}.$

↑ subring of $\mathbb{C}$ but not a field.

$\frac{1}{\pi} \notin \mathbb{Q}[\pi]$. Proof by contradiction: If $\frac{1}{\pi} = a_0 + a_1\pi + a_2\pi^2 + \dots + a_n\pi^n$ with $n \geq 0, a_0, \dots, a_n \in \mathbb{Q}$

then $1 = a_0\pi + a_1\pi^2 + \dots + a_n\pi^{n+1}$

$$0 = -1 + a_0\pi + a_1\pi^2 + \dots + a_n\pi^{n+1}$$

π is then a root of a nonzero polynomial $-1 + a_0x + a_1x^2 + \dots + a_nx^{n+1} \in \mathbb{Q}[x]$.

$$\mathbb{Q}(\pi) = \left\{ \frac{\alpha}{\beta} : \alpha, \beta \in \mathbb{Q}[\pi], \beta \neq 0 \right\} = \left\{ \frac{a_0 + a_1\pi + a_2\pi^2 + \dots + a_n\pi^n}{b_0 + b_1\pi + b_2\pi^2 + \dots + b_m\pi^m} : a_0, \dots, a_n; b_0, \dots, b_m \in \mathbb{Q}; m, n \geq 0; b_0, \dots, b_m \text{ not all zero} \right\}.$$

$\mathbb{Q}(x) = \{ \text{rational functions in } x \text{ with rational coefficients} \}$
coefficients in $\mathbb{Q}$.

$\mathbb{Q}[x] = \{ \text{polynomials in } x \text{ with rational coefficients} \}$

Ring $\mathbb{Z} \rightarrow \mathbb{Q}$
Field

$\mathbb{Q}[x] \rightarrow \mathbb{Q}(x)$

$\mathbb{Q}[\pi] \rightarrow \mathbb{Q}(\pi)$

$\mathbb{Q}[\sqrt{2}] \rightarrow \mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}]$

$\mathbb{Q}[e] \rightarrow \mathbb{Q}(e)$

$\mathbb{Q}(e) \cong \mathbb{Q}(x) \cong \mathbb{Q}(\pi)$ but not equal

The function $\phi: \mathbb{Q}(x) \rightarrow \mathbb{Q}(\pi)$

$\phi(f(x)) = f(\pi)$ evaluate at π

$$\phi(f(x) + g(x)) = f(\pi) + g(\pi) = \phi(f(x)) + \phi(g(x))$$

$$\phi(f(x)g(x)) = f(\pi)g(\pi) = \phi(f(x))\phi(g(x))$$

ϕ is bijective

so ϕ is an isomorphism.

This works because e and π are transcendental.

We know that e and π are irrational.

What about $e+\pi$ or $e\pi$?

We guess that both $e+\pi$ and $e\pi$ are irrational but we don't know.

If α, β are irrational then can $\alpha+\beta$ be rational? Yes.

Can $\alpha\beta$ be rational? Yes. (eg. $\sqrt{2} + (3-\sqrt{2}) = 3$)

eg. $\sqrt{2} \cdot \frac{3}{\sqrt{2}} = 3$

Can α^{β} be rational? Try $\sqrt{2}^{\sqrt{2}}$.

Case (I). $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$. In this case we're done.

Case (II). $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$. Then $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$.

Either way we see the answer is true.
(This is an example of a non-constructive proof.)

Actually it is known that $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$ but this is not easy to prove.