Field Theory

Book 1

Informally, a field is a "number system" in which we can add, subtract, multiply, and divide.

Eg. $\mathbb{R} = \{\text{real numbers}\}$ eg. $\pi \in \mathbb{R}$, $\sqrt{2} \in \mathbb{R}$, $i \notin \mathbb{R}$, $7 \in \mathbb{R}$

$\mathbb{Q} = \{\text{rational numbers}\}$ $\frac{3}{5} \in \mathbb{Q}$, $7 \in \mathbb{Q}$

$\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are fields

$\mathbb{C} = \{\text{complex numbers}\} = \{a+bi : a,b \in \mathbb{R}\}$, $i = \sqrt{-1}$

$5 \times \boxed{} = 3$ solution is $\frac{3}{5} \in \mathbb{Q}$.

$\mathbb{Z} = \{\text{integers}\} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is not a field. It is a ring.

$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field.

eg. $\alpha = 3+\sqrt{2}$, $\beta = 7 - 3\sqrt{2}$ in $\mathbb{Q}[\sqrt{2}]$

$\alpha + \beta = 10 - 2\sqrt{2}$

$\alpha - \beta = -4 + 4\sqrt{2}$

$\alpha\beta = (3+\sqrt{2})(7-3\sqrt{2}) = 21 - 9\sqrt{2} + 7\sqrt{2} - 6 = 15 - 2\sqrt{2}$

$$\frac{\alpha}{\beta} = \frac{3+\sqrt{2}}{7-3\sqrt{2}} \cdot \frac{7+3\sqrt{2}}{7+3\sqrt{2}} = \frac{21 + 9\sqrt{2} + 7\sqrt{2} + 6}{49 - 18} = \frac{27 + 16\sqrt{2}}{31} = \frac{27}{31} + \frac{16}{31}\sqrt{2}$$

Similar: $\mathbb{R}[x]$ is the ring of all polynomials in x with coefficients in $\mathbb{R}$

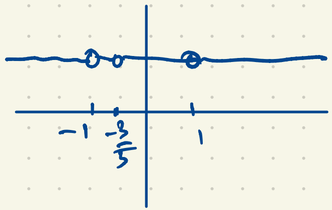
eg. $5x^2 + \pi x + \sqrt{2} \in \mathbb{R}[x]$.

This is not a field; we cannot divide $5x+3$ by $x^2 - 1$ in $\mathbb{R}[x]$ i.e. $(x^2-1) \times \boxed{?} = 5x+3$

The unique solution to this division problem is $\frac{5x+3}{x^2-1} \in \mathbb{R}(x) = \{\text{rational functions in } x \text{ with coefficients in } \mathbb{R}\}$

$$= \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{R}[x],\ g(x) \neq 0 \right\}$$

In $\mathbb{R}(x)$, $\frac{5x+3}{x^2-1} \cdot \frac{x^2-1}{5x+3} = 1$



Fields

Let F be a set containing distinct elements called 0 and 1 (thus $0 \neq 1$). Suppose addition, subtraction, multiplication and division are defined for all elements of F (except division by 0 is not defined).
Thus $a+b,\ a-b,\ ab,\ \frac{a}{d} \in F$ whenever $a, b, d \in F$ and $d \neq 0$.
Define $-a = 0 - a$.

If the following properties are satisfied by *all* elements $a, b, c, d \in F$ with $d \neq 0$, then F is a field.

$$a + b = b + a \qquad a + (b + c) = (a + b) + c \qquad ab = ba$$

$$a + 0 = a \qquad a(bc) = (ab)c \qquad 1a = a$$

$$a + (-a) = 0 \qquad a(b + c) = ab + ac \qquad \frac{a}{d}d = a$$

$$a + (-b) = a - b$$