Field Theory

Book 1

Informally, a field is a "number system" in which we can add, subtract, multiply, and divide.

Eg. $\mathbb{R} = \{\text{real numbers}\}$ eg. $\pi \in \mathbb{R}$, $\sqrt{2} \in \mathbb{R}$, $i \notin \mathbb{R}$, $7 \in \mathbb{R}$

$\mathbb{Q} = \{\text{rational numbers}\}$ $\frac{3}{5} \in \mathbb{Q}$, $7 \in \mathbb{Q}$ $\quad\quad$ $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are fields

$\mathbb{C} = \{\text{complex numbers}\} = \{a+bi : a,b \in \mathbb{R}\}$, $i = \sqrt{-1}$

$\mathbb{Z} = \{\text{integers}\} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is not a field. It is a ring.

$\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\}$ is a field.

eg. $\alpha = 3+\sqrt{2}$, $\beta = 7-3\sqrt{2}$ in $\mathbb{Q}[\sqrt{2}]$

$\alpha+\beta = 10-2\sqrt{2}$

$\alpha-\beta = -4+4\sqrt{2}$

$\alpha\beta = (3+\sqrt{2})(7-3\sqrt{2}) = 21-9\sqrt{2}+7\sqrt{2}-6 = 15-2\sqrt{2}$

$\frac{\alpha}{\beta} = \frac{3+\sqrt{2}}{7-3\sqrt{2}} \cdot \frac{7+3\sqrt{2}}{7+3\sqrt{2}} = \frac{21+9\sqrt{2}+7\sqrt{2}+6}{49-18} = \frac{27+16\sqrt{2}}{31} = \frac{27}{31} + \frac{16}{31}\sqrt{2}$