Fields

Book II

Eg. $\alpha = \sqrt{2+\sqrt{2}}$

$\alpha^2 = 2+\sqrt{2}$

$\alpha^2 - 2 = \sqrt{2}$

$\alpha^4 - 4\alpha^2 + 4 = 2$

$\alpha^4 - 4\alpha^2 + 2 = 0$

The minimal poly. of α over $\mathbb{Q}$ is $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.

(Exercise: $f(x)$ is irreducible in $\mathbb{Q}[x]$ so it really is the min. poly. of α over $\mathbb{Q}$.)

The roots of $f(x)$ are

$\alpha = \sqrt{2+\sqrt{2}}$

$-\alpha = -\sqrt{2+\sqrt{2}}$

$\beta = \sqrt{2-\sqrt{2}}$

$-\beta = -\sqrt{2-\sqrt{2}}$

$$f(x) = x^4 - 4x^2 + 2 = (x-\alpha)(x+\alpha)(x-\beta)(x+\beta)$$

In this case $E = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 + d\alpha^3 : a,b,c,d \in \mathbb{Q}\}$ contains all the roots of $f(x)$ so it is a normal extension of $\mathbb{Q}$.

$\beta = (*) + (*)\alpha + (*)\alpha^2 + (*)\alpha^3 = \alpha^3 - 3\alpha$

$\alpha\beta = \sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}} = \sqrt{4-2} = \sqrt{2} = \alpha^2 - 2$

$\Rightarrow \beta = \dfrac{\alpha^2-2}{\alpha} \in \mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$

$\beta = \alpha - \frac{2}{\alpha} = \alpha - (4\alpha - \alpha^3) = \alpha^3 - 3\alpha$

$\alpha^4 - 4\alpha^2 + 2 = 0$

$\alpha^3 - 4\alpha + \frac{2}{\alpha} = 0 \Rightarrow \frac{2}{\alpha} = 4\alpha - \alpha^3$

$\alpha^4 = 4\alpha^2 - 2$

$\alpha^6 = 4\alpha^4 - 2\alpha^2$

$= 4(4\alpha^2-2) - 2\alpha^2$

$= 14\alpha^2 - 8$

Look for an automorphism $\sigma: E \to E$ $(E = \mathbb{Q}[\alpha])$ satisfying $\sigma(\alpha) = \beta$.

$\sigma(\beta) = \sigma(\alpha^3 - 3\alpha) = \sigma(\alpha)^3 - 3\sigma(\alpha) = \beta^3 - 3\beta = (\alpha^3-3\alpha)^3 - 3(\alpha^3 - 3\alpha) = (\alpha^3-3\alpha)((\alpha^3-3\alpha)^2 - 3)$

$= (\alpha^3-3\alpha)(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 3) = (\alpha^3-3\alpha)(14\alpha^2 - 8 - 6(4\alpha^2-2) + 9\alpha^2 - 3) = (\alpha^3-3\alpha)(-\alpha^2+1) = \alpha(\alpha^2-3)(-\alpha^2+1)$

$= \alpha(-\alpha^4 + 4\alpha^2 - 3) = \alpha(-(4\alpha^2-2) + 4\alpha^2 - 3) = -\alpha$

$\sigma: \alpha \longmapsto \beta = \alpha^3 - 3\alpha \longmapsto -\alpha \longmapsto -\beta \longmapsto \alpha$

$\mathrm{Aut}\, E = \langle\sigma\rangle$ of order 4 ; cyclic.

$\sigma(\sqrt{2}) = ?$

$\sqrt{2} = \alpha\beta$

$\sigma(\sqrt{2}) = \sigma(\alpha)\sigma(\beta)$

$= \beta\cdot(-\alpha)$

$= -\alpha\beta$

$= -\sqrt{2}$

$\sigma(\alpha) = \beta$

$\sigma(-\alpha) = -\sigma(\alpha) = -\beta$

$\sigma(\beta) = -\alpha$

$\sigma(-\beta) = -\sigma(\beta) = \alpha$

Galois correspondence

$$\begin{array}{ccc}
\mathbb{Q}[\alpha] & \qquad & G = \mathrm{Aut}\,E = \langle\sigma\rangle = \{1, \sigma, \sigma^2, \sigma^3\} \\
2| & & 2| \\
\mathbb{Q}[\sqrt{2}] & & \langle\sigma^2\rangle = \{1, \sigma^2\} \\
2| & & 2| \\
\mathbb{Q} & & \langle 1 \rangle
\end{array}$$

$\sigma^2(\sqrt{2}) = \sigma(\sigma(\sqrt{2}))$
$= \sigma(-\sqrt{2}) = \sqrt{2}$.

$\sigma(\sqrt{2}) = \sigma(\alpha^2 - 2)$
$= \sigma(\alpha)^2 - 2 = \beta^2 - 2 = -\sqrt{2}$.

Compare:

$$\begin{array}{ccc} E = \mathbb{Q}[\sqrt{2}] & & G = \langle \tau \rangle \\ 2 \,\big| & & \big|\, 2 \\ \mathbb{Q} & & \langle \iota \rangle \end{array}$$

$G = \operatorname{Aut} E = \{\iota, \tau\}$, $\tau(a+b\sqrt{2}) = a - b\sqrt{2}$

Degree 2 extension : quadratic extension
" 3 " : cubic "
" 4 " : quartic "
" 5 " : quintic "

$\alpha = \sqrt[3]{2} = 2^{1/3}$

$E = \mathbb{Q}[\alpha] \supseteq \mathbb{Q}$ is an extension of degree $[E:\mathbb{Q}] = 3$ with basis $1, \underset{\sqrt[3]{2}}{\underset{\shortparallel}{\alpha}}, \alpha^2 = \sqrt[3]{4}$ $\quad (\alpha^3 = 2)$

α has min. poly. $x^3 - 2 \in \mathbb{Q}[x]$ which is irreducible

In $\mathbb{R}[x]$, $f(x) = x^3 - 2 = (x-\alpha)(x^2 + \alpha x + \alpha^2)$

Subfields

$$\begin{array}{c} E = \mathbb{Q}[\alpha] \\ \big|\, 3 \\ \mathbb{Q} \end{array}$$

If $E \supseteq F \supseteq \mathbb{Q}$ (i.e. F is an intermediate field) then the transitivity of degrees tells us $\underbrace{[E:\mathbb{Q}]}_{3} = [E:F][F:\mathbb{Q}]$

$$3 \times 1 \quad \text{or} \quad 1 \times 3$$

If $[F:\mathbb{Q}] = 1$ then $\{1\}$ is a basis for F over $\mathbb{Q}$ so $F = \{a1 : a \in \mathbb{Q}\} = \mathbb{Q}$

If $[E:F] = 1$ then (similarly) $E = F$.

More generally if $E \supseteq F$ is an extension of prime degree $p = [E:F]$ then the only intermediate extensions are E and F.

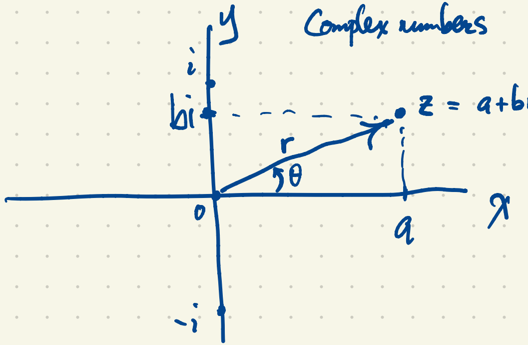
What are the automorphisms of $E = \mathbb{Q}[\alpha]$, $\alpha = \sqrt[3]{2}$? If $\phi \in \operatorname{Aut} E$ then $\phi(\alpha)^3 = \phi(\alpha^3) = \phi(2) = 2$

In $\mathbb{C}$, every poly. $f(x) \in \mathbb{C}[x]$ of degree n factors as $f(x) = a(x-r_1)(x-r_2)\cdots(x-r_n)$ $(a, r_1, r_2, \dots, r_n \in \mathbb{C})$.

eg. $x^n - 1 = (x-1)(x-\xi)(x-\xi^2)(x-\xi^3)\cdots(x-\xi^{n-1})$ where $\xi = e^{2\pi i/n}$.

de Moivre's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Complex numbers $\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$, $i = \sqrt{-1}$



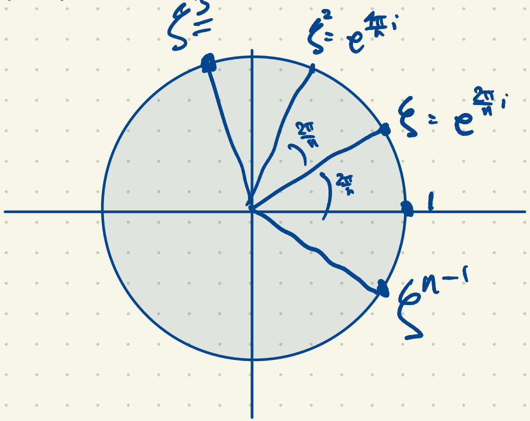
Every $z \in \mathbb{C}$ has unique representation as $z = a+bi$ $(a, b \in \mathbb{R})$ in rectangular coordinates

$a = \operatorname{Re} z$ = real part of z

$b = \operatorname{Im} z$ = imaginary part of z.

$r = |z| = \sqrt{a^2+b^2}$

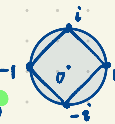
The roots of $x^n - 1$ are the n^{th} roots of unity: $1, \xi, \xi^2, \dots, \xi^{n-1}$ forming the vertices of a regular n-gon inscribed in the unit circle $|z| = 1$.



Eg. $n=4$

The fourth roots of unity are $\pm 1, \pm i$

$1, e^{\frac{i\pi}{2}} = i, e^{\frac{2i\pi}{2}} = e^{i\pi} = -1, e^{\frac{3i\pi}{2}} = -i$



Euler's Formula $e^{i\pi} = -1$

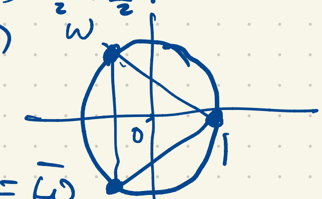
$e^{i\pi} + 1 = 0$

Eg. $n=3$: The three cube roots of unity in $\mathbb{C}$ are $1, \omega, \omega^2$ where

$\omega = e^{\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$

$\omega = \frac{-1 \pm \sqrt{-3}}{2}$



$\omega^2 = \overline{\omega}$

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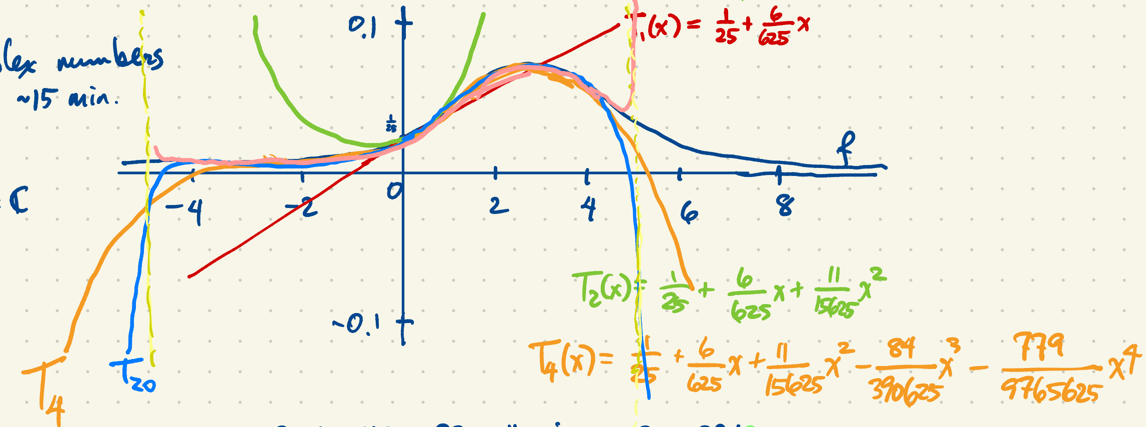
Eg. consider $f(x) = \frac{1}{x^2-6x+25}$

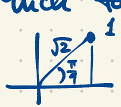
This function has poles at $x = 3 \pm 4i \in \mathbb{C}$
with $|3 \pm 4i| = 5$



By the Binomial Theorem

$$(1+i)^{11} = 1 + 11i - 55 - 165i + 330 + 462i - 462 - 330i + 165 + 55i - 11 - i = -32 + 32i$$

Much faster way to evaluate powers $z^n = (x+iy)^n = x^n + nx^{n-1}yi + \cdots + i^n y^n$ (Binomial Theorem)



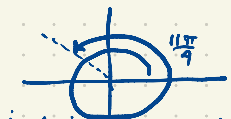
$1+i = \sqrt{2}e^{i\frac{\pi}{4}}$

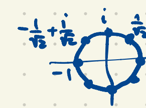
$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$(1+i)^{11} = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{11} = 32\sqrt{2}\, e^{\frac{i11\pi}{4}}$$

$$= 32\sqrt{2}\cdot\left(\frac{-1+i}{\sqrt{2}}\right)$$

$$= -32 + 32i$$





$= e^{\frac{2\pi i}{8}} = e^{\frac{\pi i}{4}}$ eighth root of unity

$\zeta = \frac{1+i}{\sqrt{2}}$

$\zeta^{11} = \zeta^3 = \frac{-1+i}{\sqrt{2}}$

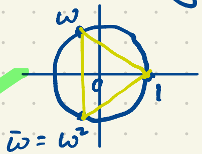
n^{th} roots of $z = re^{i\theta}$, $r = |z|$

all complex numbers whose n^{th} power is z

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}e^{\frac{i\theta}{n}},\ r^{\frac{1}{n}}e^{\frac{i\theta}{n}+\frac{2\pi i}{n}},\ r^{\frac{1}{n}}e^{\frac{i(\theta+4\pi)}{n}},\ \dots,\ r^{\frac{1}{n}}e^{i\frac{\theta+2(n-1)\pi}{n}}$$

i.e. $r^{\frac{1}{n}}e^{i\frac{\theta+2k\pi}{n}}$, $k = 0, 1, 2, \dots, n-1$

Cube roots of unity in $\mathbb{C}$: $1, \omega, \omega^2 = \bar{\omega}$

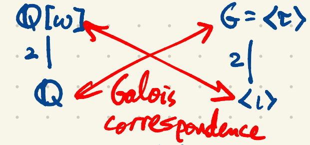


$\omega = e^{2\pi i/3} = \dfrac{-1+\sqrt{-3}}{2} = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2} i$

$\omega^2 + \omega + 1 = 0$

ω is a root of $x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$

$\tau(\omega) = \omega^2$



$G = \operatorname{Aut} \mathbb{Q}[\omega] = \langle \tau \rangle = \{\iota, \tau\}$
where $\tau(z) = \bar{z}$

Now let $\alpha = \sqrt[3]{2}$, $F = \mathbb{Q}[\alpha]$.

The min. poly. of α over $\mathbb{Q}$ is $x^3 - 2 \in \mathbb{Q}[x]$.

$F = \mathbb{Q}[\alpha] = \{ a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q} \}$, $\operatorname{Aut} F = \{\iota\}$

$3 |$

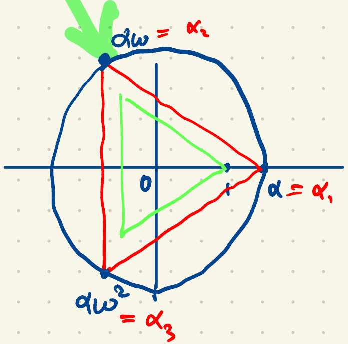
$\mathbb{Q}$

$x^3 - 2 = (x-\alpha)\underbrace{(x^2 + \alpha x + \alpha^2)}$

The other roots of x^3-2 are not in $F = \mathbb{Q}[\alpha]$ i.e. the extension $F \supset \mathbb{Q}$ is not normal.

scale by factor of α

$x^3 - 2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ where $\alpha_1 = \alpha$, $\alpha_2 = \alpha\omega$, $\alpha_3 = \alpha\omega^2$



$\alpha^3 = 2$

$(\alpha\omega)^3 = \alpha^3\omega^3 = 2 \cdot 1 = 2$

$(\alpha\omega^2)^3 = \alpha^3\omega^6 = 2 \cdot 1 = 2$

$x^3 - 2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$
$= (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$

basis $\{1, \omega\}$ so $[E:F] = 2$ basis $1, \alpha, \alpha^2$ so $[F:\mathbb{Q}] = 3$

$E \supset F \supset \mathbb{Q}$

$E = \mathbb{Q}[\alpha_1, \alpha_2, \alpha_3] = \mathbb{Q}[\alpha, \omega]$, $F = \mathbb{Q}[\alpha]$

$[E : \mathbb{Q}] = 2 \cdot 3 = 6$

$\omega = \frac{1}{2}\alpha_1^2\alpha_2$
$= \frac{1}{2} \cdot 2^{2/3} \cdot 2^{1/3}\omega$
$= \omega$

There are $3! = 6$ permutations of $\alpha_1, \alpha_2, \alpha_3$.

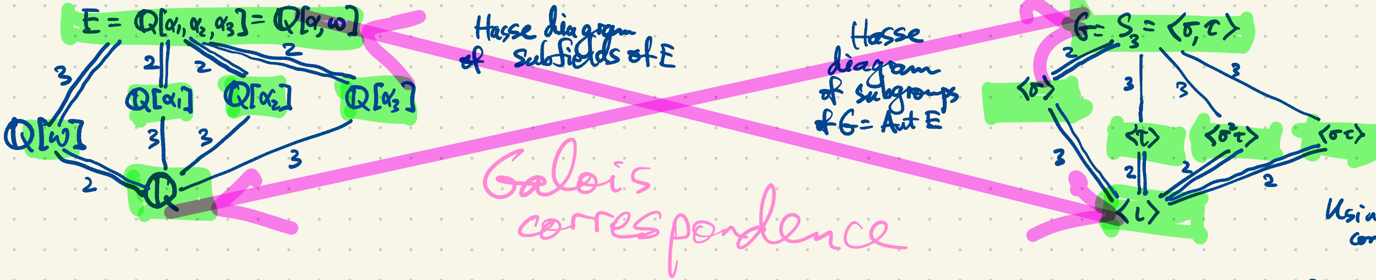
In $S_3 = \langle \sigma, \tau \rangle$, $\sigma = (123)$, $\tau = (23)$.

x	$\sigma(x)$	$\tau(x)$ ← complex conjugation
α_1	α_2	α_1
α_2	α_3	α_3
α_3	α_1	α_2
α	$\alpha\omega$	α
ω	ω	ω^2

$\sigma(\omega) = \sigma\left(\dfrac{\alpha_2}{\alpha_1}\right) = \dfrac{\sigma(\alpha_2)}{\sigma(\alpha_1)} = \dfrac{\alpha_3}{\alpha_2} = \dfrac{\alpha\omega^2}{\alpha\omega} = \omega$

$\sigma\left(\dfrac{\alpha\omega}{\alpha}\right)$

$\tau(\omega) = \tau\left(\dfrac{\alpha_2}{\alpha_1}\right) = \dfrac{\alpha_3}{\alpha_1} = \dfrac{\alpha\omega^2}{\alpha} = \omega^2 = \bar{\omega}$



$\sigma^2\tau = \tau\sigma$

$\sigma\tau = \tau\sigma^2$

Double lines indicate normality.

Using right-to-left composition

A subgroup $H \leq G$ is <u>normal</u> if its left and right cosets agree i.e. $gH = Hg$ for all $g \in G$.

$\sigma^2\tau = (132)(23) = (13)$

$\sigma\tau = (123)(23) = (12)$

$\tau = (23)$

Eg. in $G = S_3$, $H = \langle\sigma\rangle = \langle(123)\rangle$ is normal.

eg. $(12)H = (12)\{\underset{\mathrm{id}}{()},\ \underset{\sigma}{(123)},\ \underset{\sigma^2}{(132)}\} = \{(12), (23), (13)\}$

$(12)(123) = (1)(23) = (23)$

$H(12) = \{(), (123), (132)\}(12) = \{(12), (13), (23)\}$

E is the <u>splitting field</u> of

$x^3 - 2 = (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$

$E = \mathbb{Q}[\alpha, \alpha\omega, \alpha\omega^2] = \mathbb{Q}[\alpha,\omega]$

$\langle\tau\rangle$ is a subgroup of G which is <u>not</u> normal in G.

$(13)\langle\tau\rangle = (13)\{(), (23)\} = \{(13), (132)\}$

$\langle\tau\rangle(13) = \{(), (23)\}(13) = \{(13), (123)\}$

The extension $\underset{\mathbb{Q}[\alpha_1]}{\mathbb{Q}[\alpha]} \supset \mathbb{Q}$ of degree $[\mathbb{Q}[\alpha]:\mathbb{Q}] = 3$ is <u>not</u> normal since the min. poly. of α over $\mathbb{Q}$ is x^3-2 with $\mathbb{Q}[\alpha]$ containing only one of the three roots of x^3-2.

In $E = \mathbb{Q}[\alpha,\omega]$ the splitting field of $x^3-2 = (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$, can we find a single element $\beta \in E$ generating E i.e. $E = \mathbb{Q}[\beta] = \{a_0 + a_1\beta + a_2\beta^2 + \cdots + a_5\beta^5 : a_0, a_1, \ldots, a_5 \in \mathbb{Q}\}$?

Such an element β must be in E but not in $\mathbb{Q}[\omega] \cup \mathbb{Q}[\alpha] \cup \mathbb{Q}[\alpha\omega] \cup \mathbb{Q}[\alpha\omega^2]$.

In a 6-dimensional vector space, we must find a vector not contained in this union of four proper subspaces of dimension 2, 3, 3, 3 respectively.

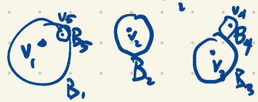
In $\mathbb{R}^3$, can $\mathbb{R}^3$ be a union of finitely many proper subspaces? No because each proper subspace of $\mathbb{R}^3$ has only dimension ≤ 2 so it covers a slice of the unit ball of volume 0. A finite union of proper subspaces covers zero volume of the unit ball; it can never cover the total volume $\frac{4}{3}\pi$ of the unit ball.

In $\mathbb{Q}^3$, i.e. points of $\mathbb{R}^3$ with rational coordinates, can $\mathbb{Q}^3 = U_1 \cup U_2 \cup U_3 \cup \cdots \cup U_k$ with $U_i \le \mathbb{Q}^3$ proper subspaces? The volume of $\mathbb{Q}^3$ (as a subset of $\mathbb{R}^3$) is zero.

$\mathbb{Q}^3 = \{v_1, v_2, v_3, v_4, \ldots\}$ is countably infinite.

Let $\varepsilon > 0$. We will show that the volume of $\mathbb{Q}^3$ is at most ε.

Take a ball B_i of radius small enough centered at v_i such that its volume is less than $\frac{\varepsilon}{2^i}$. $(i = 1,2,3,4,\ldots)$



$\bigcup_{i=1}^{\infty} B_i$ has volume $< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{16} + \cdots = \varepsilon$. Now $\mathbb{Q}^3 \subset \bigcup_{i=1}^{\infty} B_i$, so $\mathrm{Vol}(\mathbb{Q}^3) < \varepsilon$.

Try another approach. Suppose $\mathbb{Q}^3 = U_1 \cup U_2 \cup \cdots \cup U_k$, $U_i \le \mathbb{Q}^3$ proper distinct subspaces, so $\dim U_i \in \{0,1,2\}$.

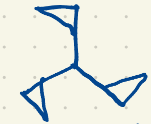
Take a line $\ell \subset \mathbb{Q}^3$ not through the origin. Then ℓ is contained in at most <u>one</u> of the subspaces U_i. With careful choice we may assume ℓ is not contained in any U_i. (Not hard.) Each U_i intersects ℓ in at most one point. This is a contradiction.

Galois theory handout: ignore "separable" for now.

Example of an extension $E \supset \mathbb{Q}$ of degree 3 with $G = \operatorname{Aut} E$ of order 3?

$f(x) = x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x]$ is irreducible

$f(x) = (x-\alpha)(x-\beta)(x-\gamma)$ where
$$\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$$
$$\alpha^3 = 1 + 2\alpha - \alpha^2$$
$$\alpha^4 = \alpha + 2\alpha^2 - \alpha^3 = \alpha + 2\alpha^2 - (1 + 2\alpha - \alpha^2) = -1 - \alpha + 3\alpha^2$$
$$\alpha^5 = 3 + 5\alpha - 4\alpha^2$$
$$\alpha^6 = -4 - 5\alpha + 9\alpha^2$$



has exactly 3 symmetries

Check that $\alpha^2 - 2$ is also a root of $f(x)$:

$f(\alpha^2-2) = (\alpha^2-2)^3 + (\alpha^2-2)^2 - 2(\alpha^2-2) - 1 = 0$ after collecting terms, so $\alpha^2 - 2 \in \{\alpha, \beta, \gamma\}$.

Can $\alpha^2 - 2 = \alpha$? No. If α is a root of $f(x) = x^3 + x^2 - 2x - 1$ and a root of $g(x) = x^2 - x - 2$ then α is a root of
$$\gcd(f(x), g(x)) = r(x)f(x) + s(x)g(x) \quad \text{by Euclid's Algorithm}$$
which is a factor of $f(x)$ of degree less than 3, a contradiction.

WLOG $\beta = \alpha^2 - 2$. Now $\beta^2 - 2$ is also a root of $f(x)$ by the same reasoning, so $\beta^2 - 2 \in \{\alpha, \beta, \gamma\}$.

As before, $\beta^2 - 2 \neq \beta$. If $\beta^2 - 2 = \alpha$ then $(\alpha^2-2)^2 - 2 = \alpha$: $\alpha^4 - 4\alpha^2 + 4 - 2 = \alpha$
$$\alpha^4 - 4\alpha^2 - \alpha + 2 = 0$$
but ~~gcd~~ $\cancel{(x^3 + x^2 - 2x - 1, x^4 - 4x^2} - x + 2) = 1$, contradiction.

Better: $-1 - \alpha + 3\alpha^2 - 4\alpha^2 - \alpha + 2 = 0$
$$1 - 2\alpha - \alpha^2 = 0$$

So $\beta^2 - 2 = \gamma$. Now $\gamma^2 - 2 = \alpha$. Indeed
$$\gamma^2 - 2 = (\beta^2-2)^2 - 2 = ((\alpha^2-2)^2 - 2)^2 - 2 = \alpha.$$

The map $x \mapsto x^2 - 2$ gives a cyclic permutation $\sigma: \alpha \mapsto \beta \mapsto \gamma \mapsto \alpha$.

The field $E = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$ of degree $[E : \mathbb{Q}] = 3$ (a cyclic cubic extension)

has automorphism group $G = \operatorname{Aut} E = \langle \sigma \rangle = \{\iota, \sigma, \sigma^2\}$, cyclic of order 3.

$$\begin{tikzcd} E=\mathbb{Q}[\alpha] \arrow[d, dash, "3"'] \arrow[dr, leftrightarrow] & G=\langle\sigma\rangle \arrow[d, dash, "3"] \arrow[dl, leftrightarrow] \\ \mathbb{Q} & \langle 1\rangle \end{tikzcd}$$

Galois Correspondence

Exercise: Find three 3×3 matrices over $\mathbb{Q}$ A, B, C which are roots of $f(x)$ satisfying

$B = A^2 - 2I$ $\qquad A^3 + A^2 - 2A - I = 0$

(not HW) $C = B^2 - 2I$ $\qquad B^3 + B^2 - 2B - I = 0$

$A = C^2 - 2I$ $\qquad C^3 + C^2 - 2C - I = 0.$

In a finite (separable) extension $E \supseteq \mathbb{Q}$ of degree $[E:\mathbb{Q}] = n$, there exists $\beta \in E$ such that $E = \mathbb{Q}[\beta]$.

(don't worry about this technical condition for now)

finite: degree $n < \infty$

(Theorem of the simple element or simple extension)

i.e. extension generated by a single element

Note: $\mathbb{C} \supset \mathbb{R}$ is a simple extension, $\mathbb{C} = \mathbb{R}[i]$.

$\mathbb{R} \supset \mathbb{Q}$ is not a simple extension. There is no $\beta \in \mathbb{R}$ satisfying $\mathbb{Q}[\beta] = \mathbb{R}$.

or $\mathbb{Q}(\beta) = \mathbb{R}$

NOVEMBER 2024

SUN	MON	TUE	WED	THU	FRI	SAT
27	28	29	30	31	1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

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Mon Nov 4
Test

Given a number field $E \supseteq \mathbb{Q}$ of degree $n = [E:\mathbb{Q}] < \infty$, there exists $\beta \in E$ such that $E = \mathbb{Q}[\beta]$ (Theorem of the Primitive Element: $E \supseteq \mathbb{Q}$ is a simple extension). It follows that

$|\mathrm{Aut}\, E| \le n$. Why? $\underbrace{1, \beta, \beta^2, \beta^3, \cdots, \beta^n}_{n+1 \text{ elements in } E \supseteq \mathbb{Q}}$ are linearly dependent so $a_0 + a_1\beta + a_2\beta^2 + \cdots + a_n\beta^n = 0$ for some $a_0, a_1, \ldots, a_n \in \mathbb{Q}$, not all zero. Actually $a_n \neq 0$, otherwise $1, \beta, \beta^2, \cdots, \beta^{n-2}$ would generate the extension, a contradiction. After dividing by $a_n \neq 0$ we get

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n \in \mathbb{Q}[x]$$

as the minimal polynomial of β. Over $\mathbb{C}$ there exist $\beta_1, \beta_2, \cdots, \beta_n \in \mathbb{C}$ such that

$\beta_1 = \beta$.

$$f(x) = (x-\beta_1)(x-\beta_2)\cdots(x-\beta_n)$$

If $\sigma \in \mathrm{Aut}\, E$ then σ must permute the n roots $\beta_1, \cdots, \beta_n$. (but $\beta_1, \cdots, \beta_n$ are not necessarily in $E = \mathbb{Q}[\beta]$.) Think of $f(x) = x^3 - 2$, $\beta = \sqrt[3]{2}$.

$$\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0 = 0$$

$$\Rightarrow \sigma(\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0) = 0$$

$$\sigma(\beta)^n + a_{n-1}\sigma(\beta)^{n-1} + \cdots + a_1\sigma(\beta) + a_0 = 0$$

$\Rightarrow$ $\sigma(\beta)$ is a root of $f(x)$.

If $\beta_1 = \beta, \beta_2, \cdots, \beta_r \in E$ and $\beta_{r+1}, \cdots, \beta_r \notin E$ then there exist automorphisms mapping $\beta = \beta_1$ to any of $\beta_1, \cdots, \beta_r$. Behind this fact is the explanation coming from the First Isomorphism for Ring Theory:

The evaluation map $\mathbb{Q}[x] \longrightarrow \mathbb{Q}[\beta]$, $g(x) \longmapsto g(\beta)$ is a homomorphism of rings. This map is onto, by definition, but not one-to-one. The kernel of this homomorphism is the principal ideal $(f(x)) = \{u(x)f(x) : u(x) \in \mathbb{Q}[x]\}$. So $\mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}[\beta] = E$

But in the same way we can evaluate at any of the $\beta_1, \dots, \beta_r \in E$ to get $\mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}[\beta_i]$ $(1 \le i \le r)$

$$\begin{tikzcd}
 & \mathbb{Q}[x]/(f(x)) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\mathbb{Q}[\beta_1] \arrow[rr, "\cong"] \arrow[d, phantom, "\parallel"] & & \mathbb{Q}[\beta_i] \arrow[d, phantom, "\parallel"] \\
E & & E
\end{tikzcd} \qquad (1 \le i \le r)$$

This gives r isomorphisms $E \to E$. $|\mathrm{Aut}(E)| = r \in \{1, 2, \dots, n\}$ where r is how many of the roots of $f(x)$ lie in $E = \mathbb{Q}[\beta]$.

When $r = n$ (all roots of $f(x)$ lie in E) then the extension $E \supseteq \mathbb{Q}$ is a <u>Galois</u> extension and we have a one-to-one Galois correspondence between subfields of E and subgroups of $G = \mathrm{Aut}\, E$.

<u>Wait</u>: what if $f(x)$ has repeated roots? Is it possible for an irreducible polynomial $f(x)$ to have repeated roots?

A field F is <u>separable</u> if every irreducible poly $f(x) \in F[x]$ has only simple roots (no multiple roots) in any extension.

<u>Theorem</u> $\mathbb{Q}$ is separable.

<u>Proof</u> Let $f(x) \in \mathbb{Q}[x]$ be irreducible of degree $n \geq 2$. If $f(x)$ has a repeated root $\alpha \in \mathbb{C}$ then $f(x) = (x-\alpha)^2 g(x)$, $g(x) \in \mathbb{C}[x]$ of degree $\geq n-2$. Then

$$f'(x) = 2(x-\alpha)g(x) + (x-\alpha)^2 g'(x) = (x-\alpha)\big(2g(x) + (x-\alpha)g'(x)\big).$$

So α is a root of

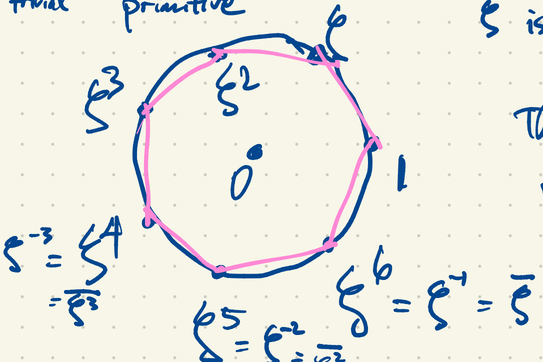
$$\gcd(\underbrace{f(x)}_{\deg f(x) = n}, \underbrace{f'(x)}_{\deg f'(x) = n-1}) = \underbrace{r(x)}_{\in\,\mathbb{Q}[x]} \underbrace{f(x)}_{\in\,\mathbb{Q}[x]} + \underbrace{s(x)}_{\in\,\mathbb{Q}[x]} \underbrace{f'(x)}_{\in\,\mathbb{Q}[x]}$$

for some $r(x), s(x) \in \mathbb{Q}[x]$.

This is a contradiction since $f(x)$ is the minimal poly. of α over $\mathbb{Q}$ whereas $\gcd(f(x), f'(x)) \in \mathbb{Q}[x]$ has degree $\le n-1$ having α as a root. $\square$

Eg. Let ξ be a primitive seventh root of unity in $\mathbb{C}$. The seventh roots of unity in $\mathbb{C}$ are

$\underbrace{1}_{\text{trivial}}, \underbrace{\xi, \xi^2, \dots, \xi^6}_{\text{primitive}}$ eg. powers of ξ^2: $1, \xi^2, \xi^4, \xi^6, \xi^8=\xi, \xi^3, \xi^5$ $\quad \xi = e^{2\pi i/7} \Rightarrow \xi^7 = e^{2\pi i} = 1$



ξ is a root of $x^7 - 1 = (x-1)(x-\xi)(x-\xi^2)(x-\xi^3)(x-\xi^4)(x-\xi^5)(x-\xi^6)$
$= (x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$

The field $E = \mathbb{Q}[\xi]$ is an extension of $\mathbb{Q}$ of degree $[E:\mathbb{Q}] = 6$
where $m(x) = x^6+x^5+x^4+x^3+x^2+x+1 \in \mathbb{Q}[x]$ is irreducible (details later).
What is $G = \operatorname{Aut} E$? and what are the subfields of E?
All roots of $m(x)$ are in $E = \mathbb{Q}[\xi]$ since they're all powers of ξ.
For each $j \in \{1,2,3,4,5,6\}$ there is an automorphism $\underline{\xi \mapsto \xi^j}$.
This gives $|G| = 6$. In fact G is cyclic.

If $\sigma \in G$ is defined by $\sigma(\xi) = \xi^2$, this defines an automorphism of E but this doesn't generate G.

$G = \{\overset{1}{\overset{\shortparallel}{\sigma_1}}, \overset{\sigma^2}{\overset{\shortparallel}{\sigma_2}}, \overset{\sigma}{\overset{\shortparallel}{\sigma_3}}, \overset{\sigma^4}{\overset{\shortparallel}{\sigma_4}}, \overset{\sigma^5}{\overset{\shortparallel}{\sigma_5}}, \overset{\sigma^3}{\overset{\shortparallel}{\sigma_6}}\}$ where $\sigma_j(\xi) = \xi^j$. So $\sigma_1(\xi) = \xi^1 = \xi$ so $\sigma_1 = 1$.

$\sigma_2(\xi) = \xi^2$
$\sigma_2^2(\xi) = \sigma_2(\sigma_2(\xi)) = \sigma_2(\xi^2) = \sigma_2(\xi)^2 = (\xi^2)^2 = \xi^4$ i.e. $\sigma_2^2 = \sigma_4$
$\sigma_2^3(\xi) = \sigma_2(\sigma_2(\sigma_2(\xi))) = \sigma_2(\xi^4) = \sigma_2(\xi)^4 = (\xi^2)^4 = \xi^8 = \xi$ i.e. $\sigma_2^3 = 1$
$\langle \sigma_2 \rangle = \{\overset{\sigma_1}{1}, \sigma_2, \sigma_4\}$

$\sigma_3(\xi) = \xi^3$
$\sigma_3^2(\xi) = \sigma_3(\sigma_3(\xi)) = \sigma_3(\xi^3) = \sigma_3(\xi)^3 = (\xi^3)^3 = \xi^9 = \xi^2$ i.e. $\sigma_3^2 = \sigma_2$
$\sigma_3^3(\xi) = \sigma_3(\sigma_3(\sigma_3(\xi))) = \sigma_3(\xi^2) = \sigma_3(\xi)^2 = (\xi^3)^2 = \xi^6$ i.e. $\sigma_3^3 = \sigma_6$

$\sigma_3^4(\xi) = \sigma_3(\sigma_3(\sigma_3(\sigma_3(\xi)))) = \sigma_3(\xi^6) = \sigma_3(\xi)^6$
$= (\xi^3)^6 = \xi^{18} = \xi^4 \Rightarrow \sigma_3^4 = \sigma_4$
$\sigma_3^5(\xi) = \xi^5 \Rightarrow \sigma_3^5 = \sigma_5$
$\sigma_3^6 = 1$

$G = \langle \sigma_3 \rangle = \langle \sigma \rangle$ where $\sigma = \sigma_3$
$\sigma(\xi) = \xi^3$

$$\begin{tikzcd}
 & E=\mathbb{Q}[\xi] \arrow[dl, dash, "2"'] \arrow[dr, dash, "3"] & \\
\mathbb{Q}[\alpha] \arrow[dr, dash, "3"'] & & \mathbb{Q}[\theta]=\mathbb{Q}[\sqrt{-7}] \arrow[dl, dash, "2"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & G=\langle\sigma\rangle \arrow[dl, dash, "3"'] \arrow[dr, dash, "2"] & \\
\langle\sigma^3\rangle \arrow[dr, dash, "2"'] & & \langle\sigma^2\rangle \arrow[dl, dash, "3"] \\
 & \langle\iota\rangle &
\end{tikzcd}$$

$\sigma: \xi \mapsto \xi^3 \mapsto \xi^2 \mapsto \xi^6$
$\mapsto \xi^4 \mapsto \xi^5 \mapsto \xi$

What does σ^3 fix?

$\sigma^3: \xi \longleftrightarrow \xi^6, \ \xi^2 \longleftrightarrow \xi^5, \ \xi^3 \longleftrightarrow \xi^4$

$\sigma^3(\xi+\xi^6) = \xi^6+\xi$

So $\alpha = \xi+\xi^6$ is fixed by σ^3.

Since θ is in an extension $\mathbb{Q}[\theta] > \mathbb{Q}$ of degree 2, it is algebraic of degree ≤ 2.
(actually degree 2)
(θ must be a quadratic irrational)
$1, \theta, \theta^2$ are linearly dependent over $\mathbb{Q}$.

$\theta^2 = (\xi+\xi^2+\xi^4)(\xi+\xi^2+\xi^4)$
$= \xi^2+\xi^4+\xi + 2\xi^3 + 2\xi^5 + 2\xi^6$
$0 = 2+2\xi+2\xi^2+2\xi^3+2\xi^4+2\xi^5+2\xi^6$

$\theta^2 = -2-\xi-\xi^2-\xi^4 = -2-\theta$

$\theta^2+\theta+2=0$

$\theta = \dfrac{-1\pm\sqrt{1-8}}{2} = \dfrac{-1\pm\sqrt{-7}}{2} = \dfrac{-1+i\sqrt{7}}{2} = \dfrac{-1+\sqrt{-7}}{2}$

$2\theta+1 = \sqrt{-7}$
$1, \theta$ is a basis for $\mathbb{Q}[\theta]$
$1, \sqrt{-7}$ is also a basis ......

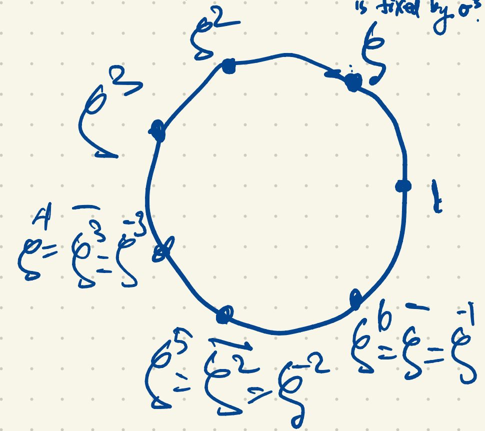
$\alpha = \xi+\xi^6$
$= \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$
$= 2\cos\frac{2\pi}{7}$

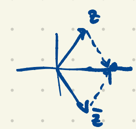
$\sigma^2: \xi \mapsto \xi^2 \mapsto \xi^4 \mapsto \xi$
$\xi^3 \mapsto \xi^6 \mapsto \xi^5 \mapsto \xi^3$

$\sigma^2(\text{something}) = \text{itself}$

$\theta = \xi+\xi^2+\xi^4 \qquad = \xi^2+\xi^4+\xi$

$\sigma^2(\theta) = \theta$





$z+\bar{z} = 2\,\mathrm{Re}\, z$