Math 4520

Fields

Book 2

Polynomials over $\mathbb{F}_2 = \{0,1\} = F$ the field of order 2.

There are 2^n polynomials of degree n over $\mathbb{F}_2$:

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_0, \ldots, a_{n-1} \in \mathbb{F}_2.$$

(necessarily monic).

Polynomials $f(x) \in F[x]$

Degree 1: x, $x+1$ are both irreducible

Degree 2: $x^2 = x \cdot x$, $x^2+1 = (x+1)^2$, $x^2+x = x(x+1)$, x^2+x+1 irreducible

$= (x+1)(x+1)$ reducible

Degree 3: $x^3 + ax^2 + bx + c$, $a, b, c \in F$

Reducible:

$x^3 = x \cdot x \cdot x$

$x^3 + x^2 = x \cdot x \cdot (x+1)$

$x^3 + x = x \cdot (x+1)(x+1)$

$x^3 + x^2 + x + 1 = (x+1)(x+1)(x+1)$

$x^3 + x^2 + x = x(x^2+x+1)$

$x^3 + 1 = (x+1)(x^2+x+1)$

Irreducible: $x^3 + x + 1$, $x^3 + x^2 + 1$

Degree 4:

$x(x^3+x+1) = x^4 + x^2 + x$

$(x+1)(x^3+x+1) = x^4 + x^3 + x^2 + 1$

$x(x^3+x^2+1) = x^4 + x^3 + x$

$(x+1)(x^3+x^2+1) = x^4 + x^2 + x + 1$

$(x^2+x+1)(x^2+x+1) = x^4 + x^2 + 1$

$x^2(x^2+x+1) = x^4 + x^3 + x^2$

$(x^2+1)(x^2+x+1) = (x+1)^2(x^2+x+1) = x^4 + x^3 + x + 1$

$(x^2+x)(x^2+x+1) = x^4 + x$

x^4

$x^3(x+1) = x^4 + x^3$

$x^2(x+1)^2 = x^4 + x^2$

$x(x+1)^3 = x^4 + x^3 + x^2 + x$

$(x+1)^4 = x^4 + 1$

reducible

irreducible

$x^4 + x^3 + x^2 + x + 1$

$x^4 + x + 1$

$x^4 + x^3 + 1$

Given prime p, and integer $n \geq 1$, there exists an irreducible polynomial of degree n in $\mathbb{F}_p[x]$.

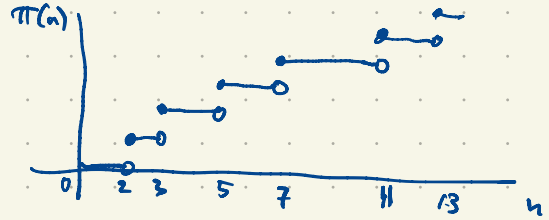
In fact we can count the number of irreducible polynomials of each degree.

$\mathbb{Z}$ and $\mathbb{F}_p[x]$ are two rings which are commutative rings with identity having unique ~~factorization~~ into irreducible factors. They are both Euclidean domains (having Division Algorithm and Euclidean algorithm).

There are roughly $\frac{n}{\ln n}$ prime numbers less than n for large n.

$\pi(n)$ = number of primes $\leq n$.

$\pi(10) = 4$, $\pi(11) = 5$.



$\pi(n) \sim \frac{n}{\ln n}$ Prime number Theorem

Gauss conjectured this formula.

$\lim_{n \to \infty} \frac{\pi(n)}{n/\ln n} = 1.$

In $F[x]$, $F = \mathbb{F}_2 = \{0,1\}$, the polynomial $x^2+x+1 \in F[x]$ is irreducible.
The extension $E = F[\alpha] = \{a+b\alpha : a,b \in F\}$ is an extension field i.e. $F \subset E$ is a subfield.
$= \{0, 1, \alpha, \underbrace{\alpha+1}_{\beta}\}$ This is the field of order 4 found previously.

$$\alpha^2 + \underbrace{\alpha+1}_{\beta} = 0 \Rightarrow \alpha^2 = -\beta = \beta$$

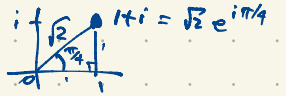
Eg. field of order 8. $m(x) = x^3+x+1 \in F[x]$, $F = \mathbb{F}_2 = \{0,1\}$. $m(x)$ is irreducible in $F[x]$.
Declare α to be a root of $m(x)$ in $E \supset F$, an extension field. $m(0) = m(1) = 1$ so m has no roots in F

$E = \{a+b\alpha+c\alpha^2 : a,b,c \in F\}$
$= \{0, 1, \alpha, \alpha+1, \alpha^2, \alpha^2+1, \alpha^2+\alpha, \alpha^2+\alpha+1\}$
$= \{0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6\}$

$m(\alpha) = \alpha^3+\alpha+1 = 0 \Rightarrow \alpha^3 = -1-\alpha = 1+\alpha$
$\alpha^4 = \alpha+\alpha^2$
$\alpha^5 = \alpha^2+\alpha^3 = \alpha^2+(\alpha+1) = \alpha^2+\alpha+1$
$\alpha^6 = \alpha^3+\alpha^2+\alpha = (\alpha+1)+\alpha^2+\alpha = \alpha^2+1$
$\alpha^7 = \alpha^3+\alpha = (\alpha+1)+\alpha = 1$.

$\beta = \alpha^2+1 = \alpha^6$
$\gamma = \alpha^2+\alpha+1 = \alpha^5$
$\beta+\gamma = (\alpha^2+1)+(\alpha^2+\alpha+1) = \alpha$
$\beta\gamma = (\alpha^2+1)(\alpha^2+\alpha+1) = \alpha^4+\alpha^3+\alpha+1 = (\alpha+\alpha^2)+(\alpha+1)+\alpha+1 = \alpha^2+\alpha$
$\beta\gamma = \alpha^6\alpha^5 = \underbrace{\alpha^{11}}_{\alpha^7\cdot\alpha^4} = \alpha^4 = \alpha^2+\alpha$.

Adding elements of E is easier using the first description $E = \{a+b\alpha+c\alpha^2 : a,b,c \in F\}$.
Multiplying elements of E is easier using the second description $E = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^6\}$.



Compare: $\mathbb{C} = \{a+bi : a,b \in \mathbb{R}\} = \{re^{i\theta} : r,\theta \in \mathbb{R}\}$

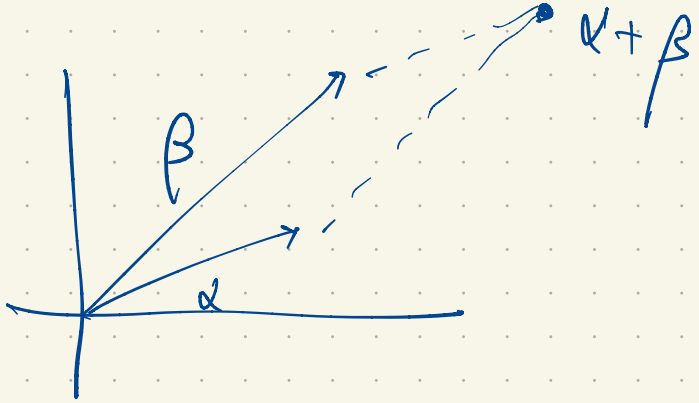
Addition in $\mathbb{C}$ is easier in rectangular form e.g. $(5+3i)+(7-11i) = 12-8i$
Multiplication in $\mathbb{C}$ ---- polar form $(re^{i\theta})(se^{i\phi}) = rs\,e^{i(\theta+\phi)}$

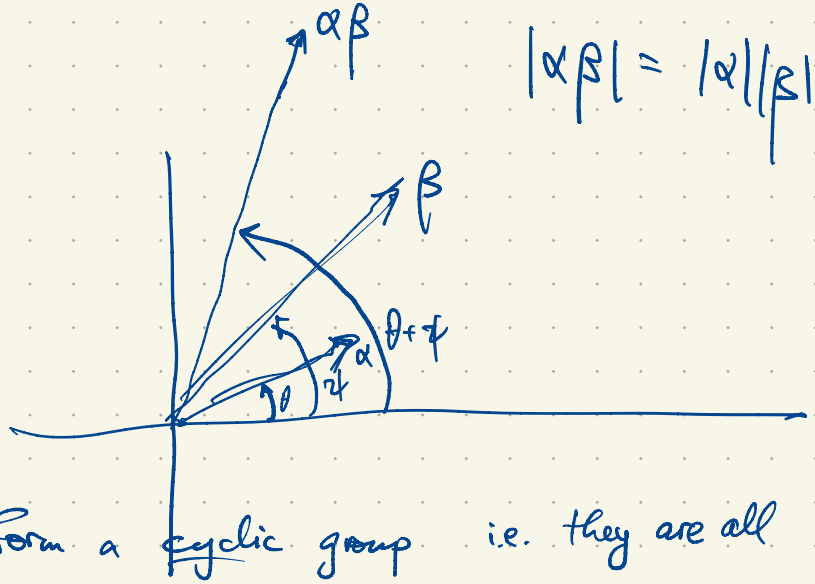


$(1+i)^{37} = (1+i)(1+i)(1+i)\cdots(1+i) = \left(\sqrt{2}\,e^{i\pi/4}\right)^{37} = \sqrt{2}^{37}\cdot e^{(i\pi/4)37} = 262144\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)$
$= -262144 - 262144i$

$\sqrt{2} = 2^{1/2}$
$\sqrt{2}^{37} = 2^{37/2} = 2^{18}\sqrt{2}$ $\quad 1024 \cdot 256$
$= 262144\sqrt{2}$

$e^{i\frac{\pi}{4}\cdot 37} = e^{i\frac{\pi}{4}\cdot 37}$ $\quad \frac{37}{4} = 9\frac{1}{4}$
$= e^{8\pi i}\cdot e^{\frac{5\pi}{4}i}$
$1 \qquad -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$





$|\alpha\beta| = |\alpha||\beta|$

In any finite field, the nonzero elements form a cyclic group i.e. they are all powers of a single element called a primitive element.

Field of order 16: $m(x) = x^4 + x + 1 \in F[x]$ is irreducible, $F = \{0,1\} = \mathbb{F}_2$.

Let α be a root of $m(x)$ i.e. $\alpha^4 + \alpha + 1 = 0$ i.e. $\alpha^4 = \alpha + 1$.

$\alpha^4 = \alpha + 1$
$\alpha^5 = \alpha^2 + \alpha$

$E = \{a + b\alpha + c\alpha^2 + d\alpha^3 : a,b,c,d \in F\}$, $|E| = 2^4 = 16$.

$= \{0, 1, \alpha, \alpha^2, \alpha^3, \cdots, \alpha^{14}\}$, $\alpha^{15} = 1$. α is primitive.

If F is any field then $F^\times = \{\text{nonzero elements in } F\}$ is an abelian (multiplicative) group.

If R is any ring with identity then $R^\times = \{\text{units in } R\}$ is a group.
(invertible elements)

eg. $R = \{n \times n \text{ matrices over } \mathbb{R}\}$ then $R^\times = GL_n(\mathbb{R}) = \{\text{invertible } n \times n \text{ matrices over } \mathbb{R}\}$

If R is a commutative ring with identity then $R^\times$ is an abelian group.

eg. $R = \mathbb{Z}/12\mathbb{Z} = \{0,1,2,\cdots,11\}$ with addition and multiplication mod 12.

$R^\times = \{1, 5, 7, 11\}$

×	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Klein four-group (not cyclic)

$x^2 - 1 \in R[x]$ has four roots in R.

If $R = \mathbb{Z}/10\mathbb{Z} = \{0,1,2,\cdots,9\}$ mod 10

$R^\times = \{1, 3, 7, 9\}$

×	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

×	1	3	9	7
1	1	3	9	7
3	3	9	7	1
9	9	7	1	3
7	7	1	3	9

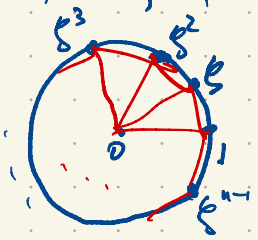
cyclic group of order 4

$\mathbb{C}^\times = \{$all nonzero complex numbers$\}$

Every finite subgroup of this subgroup is cyclic.

n^{th} roots of unity: solutions of $z^n = 1$

$\{1, \zeta, \zeta^2, \cdots, \zeta^{n-1}\}$, $n \geq 1$, $\zeta = e^{2\pi i/n}$



If F is a finite field then $F^\times$ is a cyclic group.

eg. $\mathbb{F}_7 = \{0,1,2,\cdots,6\}$, $\mathbb{F}_7^\times = \{1,2,\cdots,6\} = \langle 3 \rangle = \{1, 3, 3^2, 3^3, 3^4, 3^5\}$, $3^6 = 1$

3 is a generator for this cyclic group

$3^0 = 1$
$3^1 = 3$
$3^2 = 9 = 2$
$3^3 = 27 = 3$
$3^4 = 81 = 4$
$3^5 = 5$
$3^6 = 1$

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 8 = 1$

2 is not a generator.

3 is primitive mod 7, 2 is not.

eg. $\mathbb{F}_5 = \{0,1,2,3,4\}$, $\mathbb{F}_5^\times = \{1,2,3,4\}$ This abelian group of order 4 is cyclic which must be cyclic

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Not a Klein four-group

$\times$	1	2	4	3
1	1	2	4	3
2	2	4	3	1
4	4	3	1	2
3	3	1	2	4

cyclic group of order 4.

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 3$
$2^4 = 1$

Theorem If F is any field then every finite subgroup of $F^\times$ is cyclic.

In particular if F is any finite field, then $F^\times$ is cyclic.

Idea of proof in the case of $\mathbb{F}_5$, why can't $\mathbb{F}_5^\times = \{1,2,3,4\}$ be a Klein four-group?

In a Klein four-group, $x^2 = 1$ for all x. If $\mathbb{F}_5^\times$ were a Klein four-group then $x^2 - 1$ would have four roots 1,2,3,4. In a field, no polynomial of degree 2 can have four distinct roots.

If $f(x) = x^2 + ax + b$ has roots $r \neq s$ then $f(x) = (x-r)(x-s)$ and r, s are the only roots.

In $\mathbb{F}_{16}$, $\mathbb{F}_{16}^\times$ is an abelian group of order 15. Every element has order 1, 3, 5 or 15. If we have an element of order 15 then $\mathbb{F}_{16}^\times$ is cyclic. If every element has order 1 or 5 then $x^5 - 1$ would have 15 roots on the field.

In $F[x]$, F a field, every root $r \in F$, $f(r) = 0$ (r root of $f(x)$) $\iff$ $x - r$ is a factor of $f(x)$.
This uses the Division Algorithm. $f(x) = (x-r)q(x) + h(x)$. When r is a root, $h = 0$ so $f(x) = (x-r)q(x)$.
$\deg h(x) < 1$
$h = h(x) =$ constant

If $r_1, \dots, r_n$ are distinct roots then $f(x) = (x-r_1)(x-r_2)\cdots(x-r_n)q(x) \Rightarrow n \le \deg f(x)$.

Oct

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2	4	(6) Test

Application: to cryptography

eg. $\mathbb{F}_{13} = \{0, 1, 2, \dots, 12\}$

$\mathbb{F}_{13}^\times = \{1, 2, \dots, 12\} = \{2^k : 0 \le k \le 11\}$

k	2^k in $\mathbb{F}_{13}^\times$
0	1
1	2
2	4
3	8
4	3
5	6
6	12
7	11
8	9
9	5
10	10
11	7
12	1

2 generates the cyclic group $\mathbb{F}_{13}^\times$ of order 12.
We call 2 a primitive element.

$\gcd(k, 12) = 1$
$\iff$ k is relatively prime with 12
$\iff$ $k \equiv 1, 5, 7$ or $11 \bmod 12$

$2, 6, 11, 7 \in \mathbb{F}_{13}$ are primitive
(any one of these serves as a generator for $\mathbb{F}_{13}^\times$)

Solve $2^x \equiv 9 \bmod 13$.

$x \equiv 8 \bmod 12$. $\quad x = 12k + 8$

$2^x = 2^{12k+8} = (2^{12})^k \cdot 2^8 = 1 \cdot 9 = 9$.

If p is a large prime then solving

$2^x \equiv a$ for x (when $a \in \mathbb{F}_p^\times$ is given)

or better yet, $g^x = a$ (g is a primitive element)

is HARD.

Computing 2^x or g^x is EASY.

eg. $p = 47$

$2^{46} \equiv 1 \bmod 47$ (Fermat's Little Theorem or Lagrange's Theorem)

$2^{19} = ?$

squaring: $2^1 = 2$, $2^2 = 4$, $2^4 = 16$, $2^8 = 256 = 21$, $2^{16} = 441 = 18$

$2^{19} = 2^{16} \cdot 2^2 \cdot 2^1 = 18 \cdot 4 \cdot 2 = 3$.

$2^? = 29$ Solving this is the discrete logarithm problem

In $\mathbb{R}^* = \{\text{nonzero real numbers}\}$ solve $2^x = 29 \iff x = \log_2 29 = \frac{\ln 29}{\ln 2}$

$\ln 2^x = \ln 29$

$x \cdot \ln 2 = \ln 29$

$x = \frac{\ln 29}{\ln 2}$

Diffie-Hellman Key Exchange Protocol

Alice and Bob are able to communicate over an insecure channel (eg. email)
They need to share a secret key eg. (long, alphanumeric string).
Alice generates a large prime p. (eg. 250 digits long)
She also chooses a random $g \in \mathbb{F}_p^*$ i.e. $g \in \{1, 2, \cdots, p-1\}$.
Alice sends g and p to Bob. (This is safe.)
Alice chooses $a \in \{2, 3, \cdots, p-2\}$ randomly and keeps this a secret.

Bob chooses $b \in \{2, 3, \cdots, p-2\}$ ... ... — — — — .

Alice computes $g^a \bmod p$ (in $\mathbb{F}_p$) and sends it to Bob.
Bob ... $g^b \bmod p$ (in $\mathbb{F}_p$) ... ... Alice.

Alice computes $(g^b)^a = g^{ab} \bmod p$.
Bob computes $(g^a)^b = g^{ab} \bmod p$.
} This secret value is shared by Alice and Bob.

An eavesdropper knows g, p, g^a, g^b. The eavesdropper does not know a, b, g^{ab} (in $\mathbb{F}_p$).

integers mod p.

How can an eavesdropper figure out $g^{ab} \bmod p$ from g, p, g^a, g^b? If the eavesdropper can figure out a or b then it's easy to get g^{ab}. Is there a shortcut to finding g^{ab} without first finding a or b? Maybe but probably not.

Let F be a field.

$F[x]$ is the ring of all polynomials in x with coefficients in F.

$F[x]$ is an integral domain: commutative ring with identity 1 having no zero divisors.

So we can form the field of quotients $F(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in F[x], \ g(x) \neq 0 \right\}$.

(just like how we construct $\mathbb{Q}$ from $\mathbb{Z}$.)

x is just a symbol, not a number.

$F[[x]] = \{$ all power series in x with coefficients in $F \}$

$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots : a_i \in F \}$

$= \left\{ \sum_{n=0}^{\infty} a_n x^n : a_n \in F \right\}$

$F[x] \subset F[[x]]$ subring

$1 + 3x - 5x^2 = 1 + 3x - 5x^2 + 0x^3 + 0x^4 + 0x^5 + \cdots$

$= -5x^2 + 3x + 1$

If $f(x) = 3 + 2x - x^2 - 7x^3 + \cdots$

$g(x) = -1 + x^2 + 11x^3 + \cdots$

$f(x) + g(x) = 2 + 2x + 4x^3 + \cdots$

$f(x) - g(x) = 4 + 2x - 2x^2 - 18x^3 + \cdots$

$f(x)g(x) = (3 + 2x - x^2 - 7x^3 + \cdots)(-1 + x^2 + 11x^3 + \cdots) = -3 - 2x + 4x^2 + 42x^3 + \cdots$

$(1 - x)(1 + x + x^2 + x^3 + \cdots) = 1$

$1 + 2 + 2^2 + 2^4 + 2^5 + 2^6 + \cdots = -1$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

If we allow $f(x) \sum_{n=-\infty}^{\infty} a_n x^n = \cdots + a_{-3}x^{-3} + a_{-2}x^{-2} + a_{-1}x^{-1} + a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$g(x) = \cdots + b_{-3}x^{-3} + b_{-2}x^{-2} + b_{-1}x^{-1} + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$

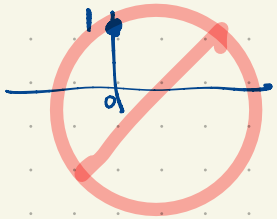
then $f(x)g(x)$ is undefined in general.

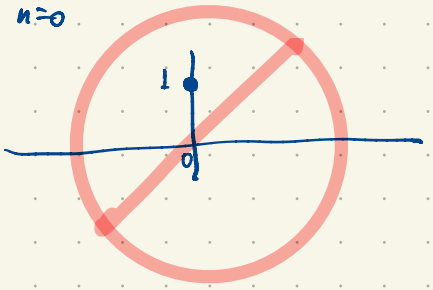
eg. in $\mathbb{R}[[x]]$ consider $f(x) \in \mathbb{R}[[x]]$ defined by $f(x) = 1 + x + 2x^2 + 6x^3 + 24x^4 + 120x^5 + 720x^6 + \cdots$

$$= \sum_{n=0}^{\infty} n!\, x^n .$$

$$f(x)^2 = (1 + x + 2x^2 + 6x^3 + \cdots)(1 + x + 2x^2 + 6x^3 + \cdots)$$

$$= 1 + 2x + 5x^2 + 16x^3 + \cdots$$





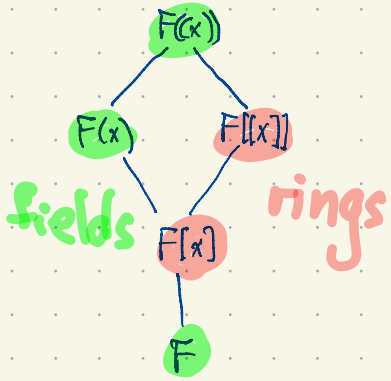
$$(x^{-1} + 1 + x + x^2 + x^3 + \cdots)(x^{-3} + 2x^{-2} - x^{-1} - 1 + 2x + 3x^2 + \cdots) = x^{-4} + 3x^{-3} + x^{-2} + x^{-1} + 3 + (*)x + (*)x^2 + \cdots$$

$F[[x]]$ is contained in a larger ring

$F((x)) = \{$ series in x allowing only finitely terms with negative exponent $\}$

$$= \left\{ \sum_{n=m}^{\infty} a_n x^n : a_n \in F \right\}, \quad m \in \mathbb{Z}$$

"Laurent series"



$$\frac{x}{x} \cdot \frac{x^{-3} + 2x^{-2} - x^{-1} - 1 + 2x + 3x^2 + \cdots}{x^{-1} + 1 + x + x^2 + x^3 + \cdots} = \frac{x^{-2} + 2x^{-1} - 1 - x + 2x^2 + 3x^3 + \cdots}{1 + x + x^2 + x^3 + x^4 + \cdots} = x^{-2} + x^{-1} - 3 + 3x^2 + \cdots$$

$$x^{-2} + 2x^{-1} - 1 - x + 2x^2 + 3x^3 + \cdots = (1 + x + x^2 + x^3 + x^4 + \cdots)(x^{-2} + x^{-1} - 3 + 3x^2 + a_3 x^3 + a_4 x^4 + \cdots)$$

$$= x^{-2} + 2x^{-1} - 1 - x + 2x^2 + \cdots$$

$$\frac{x^3}{x^3} \cdot \frac{x^{-1} + 1 + x + x^2 + x^3 + \cdots}{x^{-3} + 2x^{-2} - x^{-1} - 1 + 2x + 3x^2 + \cdots} = x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + \cdots$$

$$= \frac{x^2 + x^3 + x^4 + x^5 + x^6 + \cdots}{1 + 2x - x^2 - x^3 + 2x^4 + 3x^5 + \cdots} = x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + \cdots$$

Every rational function in $F(x)$ expands as a Laurent series.

eg. $\frac{1}{x^2-x-1} = -\frac{1}{1+x-x^2} = -1 + \underbrace{a_1}_{1}x + \underbrace{a_2}_{-2}x^2 + \underbrace{a_3}_{3}x^3 + a_4x^4 + a_5x^5 + \cdots$

Multiply both sides by $1+x-x^2$:

$$-1 = (1+x-x^2)(-1 + x - 2x^2 + 3x^3 - 5x^4 + 8x^5 + \cdots)$$

$$\frac{1}{x^2-x-1} = -1 + x - 2x^2 + 3x^3 - 5x^4 + 8x^5 - 13x^6 + 21x^7 - 34x^8 + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$$

Square both sides or take $\frac{d}{dx}$ of both sides

OR: $f(x) = \frac{1}{x^2-x-1}$

Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots$

$= -1 + x - 2x^2 + 3x^3 - \cdots$

$$f'(x) = \frac{(x^2-x-1)\cdot 0 - 1\cdot(2x-1)}{(x^2-x-1)^2} = \frac{1-2x}{(x^2-x-1)^2}$$

$f''(x) = \cdots$ $\quad f'''(x) = \cdots$