
Fields
Book II

Eg. $\alpha = \sqrt{2+\sqrt{2}}$

$\alpha^2 = 2+\sqrt{2}$

$\alpha^2 - 2 = \sqrt{2}$

$\alpha^4 - 4\alpha^2 + 4 = 2$

$\alpha^4 - 4\alpha^2 + 2 = 0$

The minimal poly. of α over $\mathbb{Q}$ is $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.

(Exercise: $f(x)$ is irreducible in $\mathbb{Q}[x]$ so it really is the min. poly. of α over $\mathbb{Q}$)

The roots of $f(x)$ are

$\alpha = \sqrt{2+\sqrt{2}}$

$-\alpha = -\sqrt{2+\sqrt{2}}$

$\beta = \sqrt{2-\sqrt{2}}$

$-\beta = -\sqrt{2-\sqrt{2}}$

$$f(x) = x^4 - 4x^2 + 2 = (x-\alpha)(x+\alpha)(x-\beta)(x+\beta)$$

In this case $E = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 + d\alpha^3 : a,b,c,d \in \mathbb{Q}\}$ contains all the roots of $f(x)$ so it is a normal extension of $\mathbb{Q}$.

$\beta = (*) + (*)\alpha + (*)\alpha^2 + (*)\alpha^3 = \alpha^3 - 3\alpha$

$\alpha\beta = \sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}} = \sqrt{4-2} = \sqrt{2} = \alpha^2 - 2$

$\Rightarrow \beta = \frac{\alpha^2-2}{\alpha} \in \mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$

$\beta = \alpha - \frac{2}{\alpha} = \alpha - (4\alpha - \alpha^3) = \alpha^3 - 3\alpha$

$\alpha^4 - 4\alpha^2 + 2 = 0$

$\alpha^3 - 4\alpha + \frac{2}{\alpha} = 0 \Rightarrow \frac{2}{\alpha} = 4\alpha - \alpha^3$

$\alpha^4 = 4\alpha^2 - 2$

$\alpha^6 = 4\alpha^4 - 2\alpha^2 = 4(4\alpha^2-2) - 2\alpha^2 = 14\alpha^2 - 8$

Look for an automorphism $\sigma : E \to E$ $(E = \mathbb{Q}[\alpha])$ satisfying $\sigma(\alpha) = \beta$.

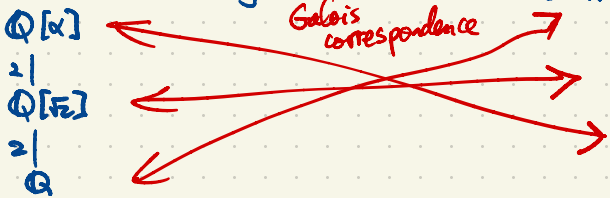
$$\sigma(\beta) = \sigma(\alpha^3 - 3\alpha) = \sigma(\alpha)^3 - 3\sigma(\alpha) = \beta^3 - 3\beta = (\alpha^3-3\alpha)^3 - 3(\alpha^3-3\alpha) = (\alpha^3-3\alpha)((\alpha^3-3\alpha)^2 - 3)$$

$$= (\alpha^3-3\alpha)(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 3) = (\alpha^3-3\alpha)(14\alpha^2 - 8 - 6(4\alpha^2-2) + 9\alpha^2 - 3) = (\alpha^3-3\alpha)(-\alpha^2+1) = \alpha(\alpha^2-3)(-\alpha^2+1)$$

$$= \alpha(-\alpha^4 + 4\alpha^2 - 3) = \alpha(-(4\alpha^2-2) + 4\alpha^2 - 3) = -\alpha$$

$\sigma : \alpha \longmapsto \beta = \alpha^3 - 3\alpha \longmapsto -\alpha \longmapsto -\beta \longmapsto \alpha$

$\mathrm{Aut}\, E = \langle\sigma\rangle$ of order 4; cyclic.



$\sigma(\sqrt{2}) = ?$

$\sqrt{2} = \alpha\beta$

$\sigma(\sqrt{2}) = \sigma(\alpha)\sigma(\beta)$

$= \beta \cdot (-\alpha)$

$= -\alpha\beta$

$= -\sqrt{2}$

$\sigma(\alpha) = \beta$

$\sigma(-\alpha) = -\sigma(\alpha) = -\beta$

$\sigma(\beta) = -\alpha$

$\sigma(-\beta) = -\sigma(\beta) = -(-\alpha) = \alpha$

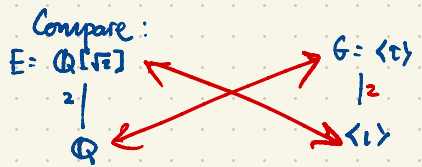
$\sigma(\sqrt{2}) = \sigma(\alpha^2 - 2)$

$= \sigma(\alpha)^2 - 2 = \beta^2 - 2 = -\sqrt{2}$.

$\sigma^2(\sqrt{2}) = \sigma(\sigma(\sqrt{2}))$

$= \sigma(-\sqrt{2}) = \sqrt{2}$.

Compare:



$G = \mathrm{Aut}\, E = \{\iota, \tau\}$, $\tau(a+b\sqrt{2}) = a-b\sqrt{2}$

Degree 2 extension : quadratic extension
" 3 " : cubic "
" 4 " : quartic "
" 5 " : quintic "

$\alpha = \sqrt[3]{2} = 2^{1/3}$

$E = \mathbb{Q}[\alpha] \supseteq \mathbb{Q}$ is an extension of degree $[E:\mathbb{Q}] = 3$

with basis $1, \alpha, \alpha^2 = \sqrt[3]{4}$ ($\alpha^3 = 2$)

α has min. poly. $x^3 - 2 \in \mathbb{Q}[x]$ which is irreducible

In $\mathbb{R}[x]$, $f(x) = x^3 - 2 = (x-\alpha)(x^2 + \alpha x + \alpha^2)$

Subfields

$E = \mathbb{Q}[\alpha]$
|3
$\mathbb{Q}$

If $E \supseteq F \supseteq \mathbb{Q}$ (i.e. F is an intermediate field) then the transitivity of degrees tells us $[E:\mathbb{Q}] = [E:F][F:\mathbb{Q}]$

3 = 3 × 1 or 1 × 3

If $[F:\mathbb{Q}] = 1$ then $\{1\}$ is a basis for F over $\mathbb{Q}$ so $F = \{a1 : a \in \mathbb{Q}\} = \mathbb{Q}$

If $[E:F] = 1$ then (similarly) $E = F$.

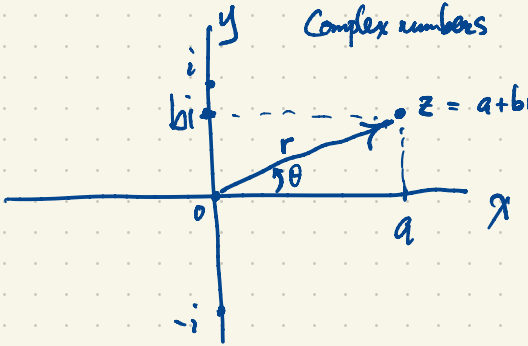
More generally if $E \supseteq F$ is an extension of prime degree $p = [E:F]$ then the only intermediate extensions are E and F.

What are the automorphisms of $E = \mathbb{Q}[\alpha]$, $\alpha = \sqrt[3]{2}$? If $\phi \in \mathrm{Aut}\, E$ then $\phi(\alpha)^3 = \phi(\alpha^3) = \phi(2) = 2$

In $\mathbb{C}$, every poly. $f(x) \in \mathbb{C}[x]$ of degree n factors as $f(x) = a(x-r_1)(x-r_2)\cdots(x-r_n)$ $(a, r_1, r_2, \cdots, r_n \in \mathbb{C})$.

eg. $x^n - 1 = (x-1)(x-\xi)(x-\xi^2)(x-\xi^3)\cdots(x-\xi^{n-1})$ where $\xi = e^{2\pi i/n}$.

de Moivre's formula: $e^{i\theta} = \cos\theta + i\sin\theta$



Complex numbers $\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$, $i = \sqrt{-1}$

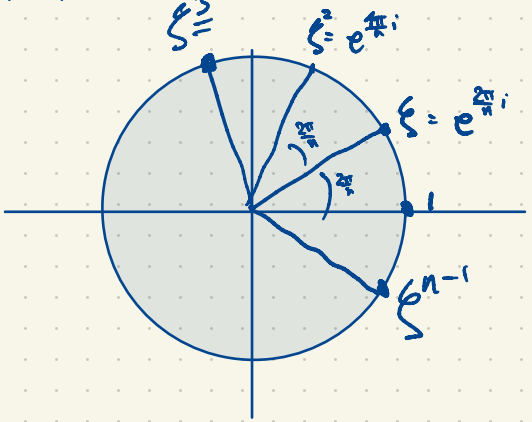
Every $z \in \mathbb{C}$ has unique representation as $z = a+bi$ $(a, b \in \mathbb{R})$ in rectangular coordinates

$a = \mathrm{Re}\, z$ = real part of z

$b = \mathrm{Im}\, z$ = imaginary part of z.

$r = |z| = \sqrt{a^2+b^2}$

The roots of $x^n - 1$ are the n^{th} roots of unity: $1, \xi, \xi^2, \cdots, \xi^{n-1}$ forming the vertices of a regular n-gon inscribed in the unit circle $|z| = 1$.



Eg. $n=4$

The fourth roots of unity are $\pm 1, \pm i$

$1, e^{i\pi/2} = i, e^{2i\pi/2} = e^{i\pi} = -1, e^{3i\pi/2} = -i$

Euler's Formula $e^{i\pi} = -1$

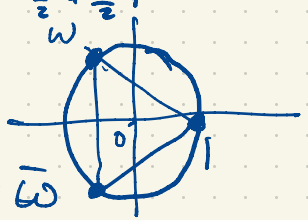
$e^{i\pi} + 1 = 0$

Eg. $n=3$: The three cube roots of unity in $\mathbb{C}$ are $1, \omega, \omega^2$ where

$\omega = e^{\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$

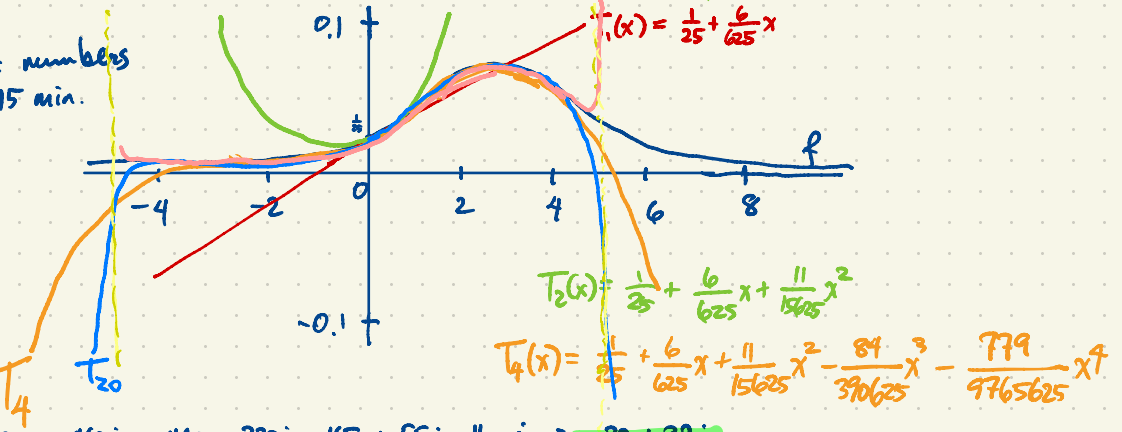
$\omega = \frac{-1 \pm \sqrt{-3}}{2}$



Follow links on course website
instructional videos → complex numbers ~15 min.

Eg. consider $f(x) = \frac{1}{x^2-6x+25}$

This function has poles at $x = 3 \pm 4i \in \mathbb{C}$
with $|3 \pm 4i| = 5$



By the Binomial Theorem

$(1+i)^{11} = 1 + 11i - 55 - 165i + 330 + 462i - 462 - 330i + 165 + 55i - 11 - i = -32 + 32i$

Much faster way to evaluate powers $z^n = (x+iy)^n = x^n + nx^{n-1}yi + \cdots + i^n y^n$ (Binomial Theorem)

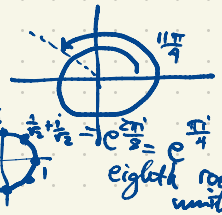
$1+i = \sqrt{2}e^{i\frac{\pi}{4}}$

$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$(1+i)^{11} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{11} = 32\sqrt{2}\, e^{i\frac{11\pi}{4}}$

$= 32\sqrt{2}\cdot\left(\frac{-1+i}{\sqrt{2}}\right)$

$= -32 + 32i$



n^{th} roots of $z = re^{i\theta}$, $r = |z|$

all complex numbers whose n^{th} power is z

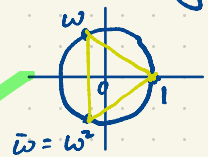
$z^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\frac{\theta}{n}},\ r^{\frac{1}{n}}e^{i\frac{\theta}{n}+\frac{2\pi i}{n}},\ r^{\frac{1}{n}}e^{i\frac{(\theta+4\pi)}{n}},\ \cdots,\ r^{\frac{1}{n}}e^{i\frac{\theta+2(n-1)\pi}{n}}$

i.e. $r^{\frac{1}{n}}e^{i\frac{\theta+2k\pi}{n}}$, $k = 0, 1, 2, \cdots, n-1$

$\xi = \frac{1+i}{\sqrt{2}}$

$\xi'' = \xi^3 = \frac{-1+i}{\sqrt{2}}$

Cube roots of unity in $\mathbb{C}$: $1, \omega, \omega^2 = \bar{\omega}$

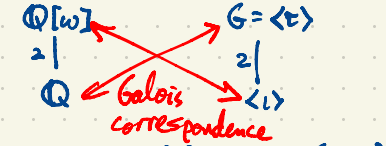


$\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\omega^2 + \omega + 1 = 0$

ω is a root of $x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$

$\tau(\omega) = \omega^2$



$G = \mathrm{Aut}\ \mathbb{Q}[\omega] = \langle \tau \rangle = \{\iota, \tau\}$ where $\tau(z) = \bar{z}$

Now let $\alpha = \sqrt[3]{2}$, $F = \mathbb{Q}[\alpha]$.

The min. poly. of α over $\mathbb{Q}$ is $x^3 - 2 \in \mathbb{Q}[x]$.

$F = \mathbb{Q}[\alpha] = \{ a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q} \}$, $\quad \mathrm{Aut}\ F = \{\iota\}$

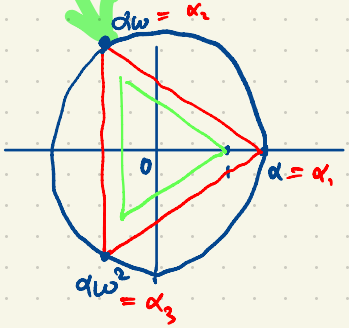
3 | ℚ

$x^3 - 2 = (x-\alpha)(x^2 + \alpha x + \alpha^2)$

The other roots of $x^3 - 2$ are not in $F = \mathbb{Q}[\alpha]$ ie. the extension $F \supset \mathbb{Q}$ is not normal.

Scale by factor of α

$x^3 - 2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ where $\alpha_1 = \alpha$, $\alpha_2 = \alpha\omega$, $\alpha_3 = \alpha\omega^2$



$\alpha^3 = 2$

$(\alpha\omega)^3 = \alpha^3\omega^3 = 2 \cdot 1 = 2$

$(\alpha\omega^2)^3 = \alpha^3\omega^6 = 2 \cdot 1 = 2$

$x^3 - 2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$

$= (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$

basis $\{1, \omega\}$ so $[E:F] = 2$; basis $1, \alpha, \alpha^2$ so $[F:\mathbb{Q}] = 3$

$E \supset F \supset \mathbb{Q}$

$\mathbb{Q}[\alpha_1, \alpha_2, \alpha_3] \quad \mathbb{Q}[\alpha]$

$\mathbb{Q}[\alpha, \omega]$

$[E:\mathbb{Q}] = 2 \cdot 3 = 6$

$\omega = \frac{1}{2}\alpha_1^2\alpha_2$

$= \frac{1}{2} \cdot 2^{2/3} \cdot 2^{1/3} \omega$

$= \omega$

There are $3! = 6$ permutations of $\alpha_1, \alpha_2, \alpha_3$.

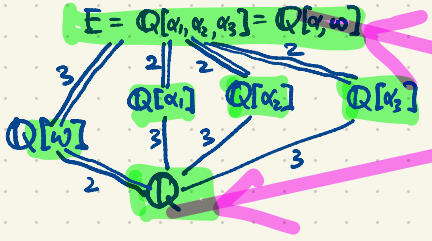
x	$\sigma(x)$	$\tau(x)$ ← complex conjugation
α_1	α_2	α_1
α_2	α_3	α_3
α_3	α_1	α_2
α	$\alpha\omega$	α
ω	ω	ω^2

In $S_3 = \langle \sigma, \tau \rangle$, $\sigma = (123)$, $\tau = (23)$.

$\sigma(\omega) = \sigma(\frac{\alpha_2}{\alpha_1}) = \frac{\sigma(\alpha_2)}{\sigma(\alpha_1)} = \frac{\alpha_3}{\alpha_2} = \frac{\alpha\omega^2}{\alpha\omega} = \omega$

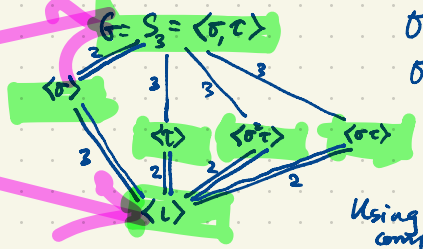
$\sigma(\frac{\alpha\omega}{\alpha})$

$\tau(\omega) = \tau(\frac{\alpha_2}{\alpha_1}) = \frac{\alpha_3}{\alpha_1} = \frac{\alpha\omega^2}{\alpha} = \omega^2 = \bar{\omega}$



Hasse diagram of subfields of E

Hasse diagram of subgroups of $G = \mathrm{Aut}\, E$



Galois correspondence

$\sigma^2\tau = \tau\sigma$

$\sigma\tau = \tau\sigma^2$

Double lines indicate normality.

Using right-to-left composition

A subgroup $H \leq G$ is normal if its left and right cosets agree i.e. $gH = Hg$ for all $g \in G$.

Eg. in $G = S_3$, $H = \langle\sigma\rangle = \langle(123)\rangle$ is normal.

eg. $(12)H = (12)\{(), (123), (132)\} = \{(12), (23), (13)\}$
($()$ = id, (123) = σ, (132) = σ^2)

$H(12) = \{(), (123), (132)\}(12) = \{(12), (13), (23)\}$

$\sigma^2\tau = (132)(23) = (13)$

$\sigma\tau = (123)(23) = (12)$

$\tau = (23)$

$(12)(123) = (1)(23) = (23)$

$\langle\tau\rangle$ is a subgroup of G which is not normal in G.

$(13)\langle\tau\rangle = (13)\{(), (23)\} = \{(13), (132)\}$

$\langle\tau\rangle(13) = \{(), (23)\}(13) = \{(13), (123)\}$

E is the splitting field of

$x^3 - 2 = (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$

$E = \mathbb{Q}[\alpha, \alpha\omega, \alpha\omega^2] = \mathbb{Q}[\alpha, \omega]$

The extension $\mathbb{Q}[\alpha] \supset \mathbb{Q}$ of degree $[\mathbb{Q}[\alpha] : \mathbb{Q}] = 3$ is not normal since the min. poly. of α over $\mathbb{Q}$ is $x^3 - 2$ with $\mathbb{Q}[\alpha]$ containing only one of the three roots of $x^3 - 2$.

In $E = \mathbb{Q}[\alpha, \omega]$ the splitting field of $x^3 - 2 = (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$, can we find a single element $\beta \in E$ generating E i.e. $E = \mathbb{Q}[\beta] = \{ q_0 + q_1\beta + q_2\beta^2 + \cdots + q_5\beta^5 : q_0, q_1, \ldots, q_5 \in \mathbb{Q} \}$?
Such an element β must be in E but not in $\mathbb{Q}[\omega] \cup \mathbb{Q}[\alpha] \cup \mathbb{Q}[\alpha\omega] \cup \mathbb{Q}[\alpha\omega^2]$.
In a 6-dimensional vector space, we must find a vector not contained in this union of four proper subspaces of dimension 2, 3, 3, 3 respectively.

In $\mathbb{R}^3$, can $\mathbb{R}^3$ be a union of finitely many proper subspaces? No because each proper subspace of $\mathbb{R}^3$ has only dimension ≤ 2 so it covers a slice of the unit ball of volume 0. A finite union of proper subspaces covers zero volume of the unit ball; it can never cover the total volume $\frac{4}{3}\pi$ of the unit ball.

In $\mathbb{Q}^3$, i.e. points of $\mathbb{R}^3$ with rational coordinates, can $\mathbb{Q}^3 = U_1 \cup U_2 \cup U_3 \cup \cdots \cup U_k$ with $U_i \leq \mathbb{Q}^3$ proper subspaces? The volume of $\mathbb{Q}^3$ (as a subset of $\mathbb{R}^3$) is zero.
$\mathbb{Q}^3 = \{ v_1, v_2, v_3, v_4, \cdots \}$ is countably infinite.
Let $\varepsilon > 0$. We will show that the volume of $\mathbb{Q}^3$ is at most ε.
Take a ball B_i of radius small enough centered at v_i such that its volume is less than $\frac{\varepsilon}{2^i}$. $(i = 1, 2, 3, 4, \ldots)$

$\bigcup_{i=1}^{\infty} B_i$ has volume $< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{16} + \cdots = \varepsilon$. Now $\mathbb{Q}^3 \subset \bigcup_{i=1}^{\infty} B_i$, so $\mathrm{Vol}(\mathbb{Q}^3) < \varepsilon$.

Try another approach. Suppose $\mathbb{Q}^3 = U_1 \cup U_2 \cup \cdots \cup U_k$, $U_i \leq \mathbb{Q}^3$ proper distinct subspaces, so $\dim U_i \in \{0, 1, 2\}$.
Take a line $l \subset \mathbb{Q}^3$ not through the origin. Then l is contained in at most one of the subspaces U_i.
With careful choice we may assume l is not contained in any U_i. (Not hard.) Each U_i intersects l in at most one point. This is a contradiction.

Galois theory handout: ignore "separable" for now.

Example of an extension $E \supset \mathbb{Q}$ of degree 3 with $G = \mathrm{Aut}\, E$ of order 3?

$f(x) = x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x]$ is irreducible

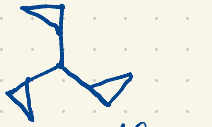
$f(x) = (x-\alpha)(x-\beta)(x-\gamma)$ where $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$

$\alpha^3 = 1 + 2\alpha - \alpha^2$

$\alpha^4 = \alpha + 2\alpha^2 - \alpha^3 = \alpha + 2\alpha^2 - (1 + 2\alpha - \alpha^2) = -1 - \alpha + 3\alpha^2$

$\alpha^5 = 3 + 5\alpha - 4\alpha^2$

$\alpha^6 = -4 - 5\alpha + 9\alpha^2$



has exactly 3 symmetries

Check that $\alpha^2 - 2$ is also a root of $f(x)$:

$f(\alpha^2-2) = (\alpha^2-2)^3 + (\alpha^2-2)^2 - 2(\alpha^2-2) - 1 = 0$ after collecting terms, so $\alpha^2 - 2 \in \{\alpha, \beta, \gamma\}$.

Can $\alpha^2 - 2 = \alpha$? No. If α is a root of $f(x) = x^3 + x^2 - 2x - 1$ and a root of $g(x) = x^2 - x - 2$ then α is a root of

$$\gcd(f(x), g(x)) = r(x) f(x) + s(x) g(x)$$ by Euclid's Algorithm

which is a factor of $f(x)$ of degree less than 3, a contradiction.

WLOG $\beta = \alpha^2 - 2$. Now $\beta^2 - 2$ is also a root of $f(x)$ by the same reasoning, so $\beta^2 - 2 \in \{\alpha, \beta, \gamma\}$.

As before, $\beta^2 - 2 \neq \beta$. If $\beta^2 - 2 = \alpha$ then $(\alpha^2-2)^2 - 2 = \alpha$: $\alpha^4 - 4\alpha^2 + 4 - 2 = \alpha$

$\alpha^4 - 4\alpha^2 - \alpha + 2 = 0$

but ~~$\gcd(x^3 + x^2 - 2x - 1,\ x^4 - 4x^2 - x + 2) = 1$~~, contradiction.

Better: $-1 - \alpha + 3\alpha^2 - 4\alpha^2 - \alpha + 2 = 0$

$1 - 2\alpha - \alpha^2 = 0$

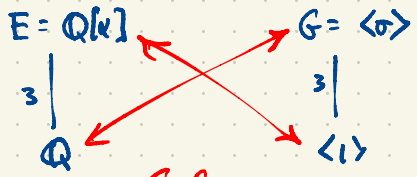
So $\beta^2 - 2 = \gamma$. Now $\gamma^2 - 2 = \alpha$. Indeed

$\gamma^2 - 2 = (\beta^2-2)^2 - 2 = ((\alpha^2-2)^2-2)^2 - 2 = \alpha$.

The map $x \mapsto x^2 - 2$ gives a cyclic permutation $\sigma: \alpha \mapsto \beta \mapsto \gamma \mapsto \alpha$.

The field $E = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$ of degree $[E : \mathbb{Q}] = 3$ (a cyclic cubic extension)

has automorphism group $G = \mathrm{Aut}\, E = \langle \sigma \rangle = \{1, \sigma, \sigma^2\}$, cyclic of order 3.



Galois Correspondence

Exercise: Find three 3×3 matrices over $\mathbb{Q}$ A, B, C which are roots of $f(x)$ satisfying

$B = A^2 - 2I$, $C = B^2 - 2I$, $A = C^2 - 2I$ (not HW)

$A^3 + A^2 - 2A - I = 0$

$B^3 + B^2 - 2B - I = 0$

$C^3 + C^2 - 2C - I = 0$.

In a finite (separable) extension $E \geq \mathbb{Q}$ of degree $[E:\mathbb{Q}] = n$, there exists $\beta \in E$ such that $E = \mathbb{Q}[\beta]$.

(don't worry about this technical condition for now)

↑ degree $n < \infty$

(Theorem of the simple element or simple extension)

i.e. extension generated by a single element

Note: $\mathbb{C} \supset \mathbb{R}$ is a simple extension, $\mathbb{C} = \mathbb{R}[i]$.

$\mathbb{R} \supset \mathbb{Q}$ is not a simple extension. There is no $\beta \in \mathbb{R}$ satisfying $\mathbb{Q}[\beta] = \mathbb{R}$, or $\mathbb{Q}(\beta) = \mathbb{R}$

Mon Nov 4
Test

Given a number field $E \supseteq \mathbb{Q}$ of degree $n = [E:\mathbb{Q}] < \infty$, there exists $\beta \in E$ such that $E = \mathbb{Q}[\beta]$ (Theorem of the Primitive Element: $E \supseteq \mathbb{Q}$ is a simple extension). It follows that $|\mathrm{Aut}\, E| \leq n$. Why? $\underbrace{1, \beta, \beta^2, \beta^3, \cdots, \beta^n}_{n+1 \text{ elements in } E \supseteq \mathbb{Q}}$ are linearly dependent so $a_0 + a_1\beta + a_2\beta^2 + \cdots + a_n\beta^n = 0$ for some $a_0, a_1, \cdots, a_n \in \mathbb{Q}$, not all zero. Actually $a_n \neq 0$, otherwise $1, \beta, \beta^2, \cdots, \beta^{n-2}$ would generate the extension, a contradiction. After dividing by $a_n \neq 0$ we get

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n \in \mathbb{Q}[x]$$

as the minimal polynomial of β. Over $\mathbb{C}$ there exist $\beta_1, \beta_2, \cdots, \beta_n \in \mathbb{C}$ such that $\beta_1 = \beta$.

$$f(x) = (x-\beta_1)(x-\beta_2)\cdots(x-\beta_n)$$

If $\sigma \in \mathrm{Aut}\, E$ then σ must permute the n roots $\beta_1, \cdots, \beta_n$. (But $\beta_1, \cdots, \beta_n$ are not necessarily in $E = \mathbb{Q}[\beta]$. Think of $f(x) = x^3 - 2$, $\beta = \sqrt[3]{2}$.

$$\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0 = 0$$

$$\Rightarrow \sigma(\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0) = 0$$

$$\sigma(\beta)^n + a_{n-1}\sigma(\beta)^{n-1} + \cdots + a_1\sigma(\beta) + a_0 = 0$$

$\Rightarrow \sigma(\beta)$ is a root of $f(x)$.

If $\beta_1 = \beta, \beta_2, \cdots, \beta_r \in E$ and $\beta_{r+1}, \cdots, \beta_n \notin E$ then there exist automorphisms mapping $\beta = \beta_1$ to any of $\beta_1, \cdots, \beta_r$.

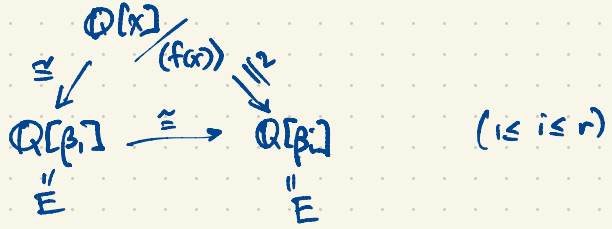
Behind this fact is the explanation coming from the First Isomorphism for Ring Theory:

The evaluation map $\mathbb{Q}[x] \longrightarrow \mathbb{Q}[\beta]$

$$g(x) \longmapsto g(\beta)$$

is a homomorphism of rings. This map is onto, by definition, but not one-to-one. The kernel of this homomorphism is the principal ideal $(f(x)) = \{u(x)f(x) : u(x) \in \mathbb{Q}[x]\}$. So $\mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}[\beta] = E$

But in the same way we can evaluate at any of the $\beta_1, \dots, \beta_r \in E$ to get $\mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}[\beta_i]$ $(1 \le i \le r)$



This gives r isomorphisms $E \to E$. $|\mathrm{Aut}(E)| = r \in \{1, 2, \dots, n\}$
where r is how many of the roots of $f(x)$ lie in $E = \mathbb{Q}[\beta]$.

When $r = n$ (all roots of $f(x)$ lie in E) then the extension $E \supseteq \mathbb{Q}$ is a Galois extension and we have a one-to-one correspondence between subfields of E and subgroups of $G = \mathrm{Aut}\, E$.

Wait: what if $f(x)$ has repeated roots? Is it possible for an irreducible polynomial $f(x)$ to have repeated roots?

A field F is separable if every irreducible poly $f(x) \in F[x]$ has only simple roots (no multiple roots) in any extension.

Theorem $\mathbb{Q}$ is separable.

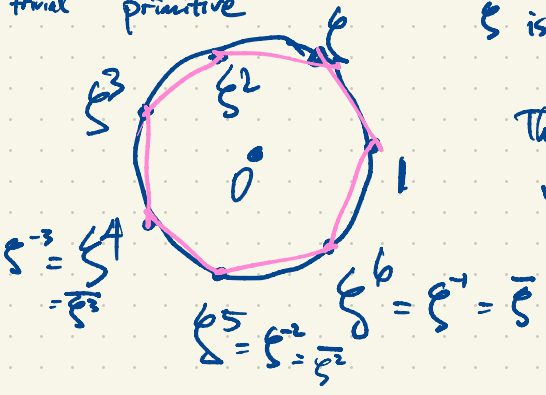
Proof Let $f(x) \in \mathbb{Q}[x]$ be irreducible of degree $n \ge 2$. If $f(x)$ has a repeated root $\alpha \in \mathbb{C}$ then
$f(x) = (x-\alpha)^2 g(x)$, $g(x) \in \mathbb{C}[x]$ of degree $\ge n-2$. Then $f'(x) = 2(x-\alpha)g(x) + (x-\alpha)^2 g'(x)$
$= (x-\alpha)(2g(x) + (x-\alpha)g'(x))$. So α is a root of $\gcd(f(x), f'(x)) = r(x)f(x) + s(x)f'(x)$ for some $r(x), s(x) \in \mathbb{Q}[x]$
($\deg f(x) = n$, $\deg f'(x) = n-1$; $\mathbb{Q}[x]$, $\mathbb{Q}[x]$, $\mathbb{Q}[x]$, $\mathbb{Q}[x]$)

This is a contradiction since $f(x)$ is the minimal poly. of α over $\mathbb{Q}$ whereas $\gcd(f(x), f'(x)) \in \mathbb{Q}[x]$ has degree $\le n-1$ having α as a root. $\square$

Eg. Let ζ be a primitive seventh root of unity in $\mathbb{C}$. The seventh roots of unity in $\mathbb{C}$ are

$1, \zeta, \zeta^2, \dots, \zeta^6$ (1 trivial; $\zeta, \dots, \zeta^6$ primitive)

e.g. powers of ζ^2: $1, \zeta^2, \zeta^4, \zeta^6, \zeta^8 = \zeta, \zeta^3, \zeta^5$ $\quad \zeta = e^{2\pi i/7} \Rightarrow \zeta^7 = e^{2\pi i} = 1$



ζ is a root of $x^7 - 1 = (x-1)(x-\zeta)(x-\zeta^2)(x-\zeta^3)(x-\zeta^4)(x-\zeta^5)(x-\zeta^6)$

$= (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

The field $E = \mathbb{Q}[\zeta]$ is an extension of $\mathbb{Q}$ of degree $[E:\mathbb{Q}] = 6$

where $m(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$ is irreducible (details later).

What is $G = \mathrm{Aut}\, E$? and what are the subfields of E?

All roots of $m(x)$ are in $E = \mathbb{Q}[\zeta]$ since they're all powers of ζ.

For each $j \in \{1, 2, 3, 4, 5, 6\}$ there is an automorphism $\zeta \longmapsto \zeta^j$.

This gives $|G| = 6$. In fact G is cyclic.

If $\sigma \in G$ is defined by $\sigma(\zeta) = \zeta^2$, this defines an automorphism of E but this doesn't generate G.

$G = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ where $\sigma_j(\zeta) = \zeta^j$. So $\sigma_1(\zeta) = \zeta^1 = \zeta$ so $\sigma_1 = 1$.

$\sigma_2(\zeta) = \zeta^2$

$\sigma_2^2(\zeta) = \sigma_2(\sigma_2(\zeta)) = \sigma_2(\zeta^2) = \sigma_2(\zeta)^2 = (\zeta^2)^2 = \zeta^4$ i.e. $\sigma_2^2 = \sigma_4$

$\sigma_2^3(\zeta) = \sigma_2(\sigma_2(\sigma_2(\zeta))) = \sigma_2(\zeta^4) = \sigma_2(\zeta)^4 = (\zeta^2)^4 = \zeta^8 = \zeta$ i.e. $\sigma_2^3 = 1$

$\sigma_3(\zeta) = \zeta^3$

$\sigma_3^2(\zeta) = \sigma_3(\sigma_3(\zeta)) = \sigma_3(\zeta^3) = \sigma_3(\zeta)^3 = (\zeta^3)^3 = \zeta^9 = \zeta^2$ i.e. $\sigma_3^2 = \sigma_2$

$\sigma_3^3(\zeta) = \sigma_3(\sigma_3(\sigma_3(\zeta))) = \sigma_3(\zeta^2) = \sigma_3(\zeta)^2 = (\zeta^3)^2 = \zeta^6$ i.e. $\sigma_3^3 = \sigma_6$

$\sigma_3^4(\zeta) = \sigma_3(\sigma_3(\sigma_3(\sigma_3(\zeta)))) = \sigma_3(\sigma_3(\sigma_3(\zeta^3))) = \sigma_3(\zeta^6) = \sigma_3(\zeta)^6$

$= (\zeta^3)^6 = \zeta^{18} = \zeta^4 \Rightarrow \sigma_3^4 = \sigma_4$

$\sigma_3^5(\zeta) = \zeta^5 \Rightarrow \sigma_3^5 = \sigma_5$

$\sigma_3^6 = 1$.

$\langle \sigma_2 \rangle = \{1, \sigma_2, \sigma_4\}$