Fields

Book II

Eg. $\alpha = \sqrt{2+\sqrt{2}}$

$\alpha^2 = 2+\sqrt{2}$

$\alpha^2 - 2 = \sqrt{2}$

$\alpha^4 - 4\alpha^2 + 4 = 2$

$\alpha^4 - 4\alpha^2 + 2 = 0$

The minimal poly. of α over $\mathbb{Q}$ is $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.

(Exercise: $f(x)$ is irreducible in $\mathbb{Q}[x]$ so it really is the min. poly. of α over $\mathbb{Q}$.)

The roots of $f(x)$ are

$\alpha = \sqrt{2+\sqrt{2}}$

$-\alpha = -\sqrt{2+\sqrt{2}}$

$\beta = \sqrt{2-\sqrt{2}}$

$-\beta = -\sqrt{2-\sqrt{2}}$

$$f(x) = x^4 - 4x^2 + 2 = (x-\alpha)(x+\alpha)(x-\beta)(x+\beta)$$

In this case $E = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 + d\alpha^3 : a,b,c,d \in \mathbb{Q}\}$ contains all the roots of $f(x)$ so it is a normal extension of $\mathbb{Q}$.

$\beta = (*) + (*)\alpha + (*)\alpha^2 + (*)\alpha^3 = \alpha^3 - 3\alpha$

$\alpha\beta = \sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}} = \sqrt{4-2} = \sqrt{2} = \alpha^2 - 2$

$\Longrightarrow \beta = \dfrac{\alpha^2-2}{\alpha} \in \mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$

$\beta = \alpha - \frac{2}{\alpha} = \alpha - (4\alpha - \alpha^3) = \alpha^3 - 3\alpha$

$\alpha^4 - 4\alpha^2 + 2 = 0$

$\alpha^3 - 4\alpha + \frac{2}{\alpha} = 0 \Rightarrow \frac{2}{\alpha} = 4\alpha - \alpha^3$

$\alpha^4 = 4\alpha^2 - 2$

$\alpha^6 = 4\alpha^4 - 2\alpha^2$

$= 4(4\alpha^2 - 2) - 2\alpha^2$

$= 14\alpha^2 - 8$

Look for an automorphism $\sigma: E \to E$ $(E = \mathbb{Q}[\alpha])$ satisfying $\sigma(\alpha) = \beta$.

$\sigma(\beta) = \sigma(\alpha^3 - 3\alpha) = \sigma(\alpha)^3 - 3\sigma(\alpha) = \beta^3 - 3\beta = (\alpha^3 - 3\alpha)^3 - 3(\alpha^3 - 3\alpha) = (\alpha^3 - 3\alpha)((\alpha^3 - 3\alpha)^2 - 3)$

$= (\alpha^3 - 3\alpha)(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 3) = (\alpha^3 - 3\alpha)(14\alpha^2 - 8 - 6(4\alpha^2 - 2) + 9\alpha^2 - 3) = (\alpha^3 - 3\alpha)(-\alpha^2 + 1) = \alpha(\alpha^2 - 3)(-\alpha^2 + 1)$

$= \alpha(-\alpha^4 + 4\alpha^2 - 3) = \alpha(-(4\alpha^2 - 2) + 4\alpha^2 - 3) = -\alpha$

$\sigma: \alpha \longmapsto \beta = \alpha^3 - 3\alpha \longmapsto -\alpha \longmapsto -\beta \longmapsto \alpha$

$\mathrm{Aut}\, E = \langle\sigma\rangle$ of order 4; cyclic.

$\sigma(\sqrt{2}) = ?$

$\sqrt{2} = \alpha\beta$

$\sigma(\sqrt{2}) = \sigma(\alpha)\sigma(\beta)$

$= \beta \cdot (-\alpha)$

$= -\alpha\beta$

$= -\sqrt{2}$

$\sigma(\alpha) = \beta$

$\sigma(-\alpha) = -\sigma(\alpha) = -\beta$

$\sigma(\beta) = -\alpha$

$\sigma(-\beta) = -\sigma(\beta) = -(-\alpha)$

$= \alpha$

Galois correspondence

$\mathbb{Q}[\alpha]$	$G = \mathrm{Aut}\, E = \langle\sigma\rangle = \{1, \sigma, \sigma^2, \sigma^3\}$
$2\mid$	$2\mid$
$\mathbb{Q}[\sqrt{2}]$	$\langle\sigma^2\rangle = \{1, \sigma^2\}$
$2\mid$	$2\mid$
$\mathbb{Q}$	$\langle 1\rangle$

$\sigma^2(\sqrt{2}) = \sigma(\sigma(\sqrt{2}))$

$= \sigma(-\sqrt{2}) = \sqrt{2}$.

$\sigma(\sqrt{2}) = \sigma(\alpha^2 - 2)$

$= \sigma(\alpha)^2 - 2 = \beta^2 - 2 = -\sqrt{2}$.

Compare:

$$\begin{tikzcd} E = \mathbb{Q}[\sqrt{2}] \arrow[d, dash, "2"'] \arrow[rrd, leftrightarrow, red] & & G = \langle \tau \rangle \arrow[d, dash, "2"] \arrow[lld, leftrightarrow, red] \\ \mathbb{Q} & & \langle \iota \rangle \end{tikzcd}$$

$G = \operatorname{Aut} E = \{\iota, \tau\}$, $\tau(a+b\sqrt{2}) = a - b\sqrt{2}$

Degree 2 extension : quadratic extension
" 3 " : cubic "
" 4 " : quartic "
" 5 " : quintic "

$\alpha = \sqrt[3]{2} = 2^{1/3}$

$E = \mathbb{Q}[\alpha] \supseteq \mathbb{Q}$ is an extension of degree $[E:\mathbb{Q}] = 3$ with basis $1, \underset{\sqrt[3]{2}}{\underset{\parallel}{\alpha}}, \alpha^2 = \sqrt[3]{4}$ $\quad (\alpha^3 = 2)$

α has min. poly. $x^3 - 2 \in \mathbb{Q}[x]$ which is irreducible

In $\mathbb{R}[x]$, $f(x) = x^3 - 2 = (x-\alpha)(x^2 + \alpha x + \alpha^2)$

Subfields

$$\begin{tikzcd} E = \mathbb{Q}[\alpha] \arrow[d, dash, "3"] \\ \mathbb{Q} \end{tikzcd}$$

If $E \supseteq F \supseteq \mathbb{Q}$ (i.e. F is an intermediate field) then the transitivity of degrees tells us $\underbrace{[E:\mathbb{Q}]}_{3} = [E:F][F:\mathbb{Q}]$

3×1 or 1×3

If $[F:\mathbb{Q}] = 1$ then $\{1\}$ is a basis for F over $\mathbb{Q}$ so $F = \{a1 : a \in \mathbb{Q}\} = \mathbb{Q}$

If $[E:F] = 1$ then (similarly) $E = F$.

More generally if $E \supseteq F$ is an extension of prime degree $p = [E:F]$ then the only intermediate extensions are E and F.

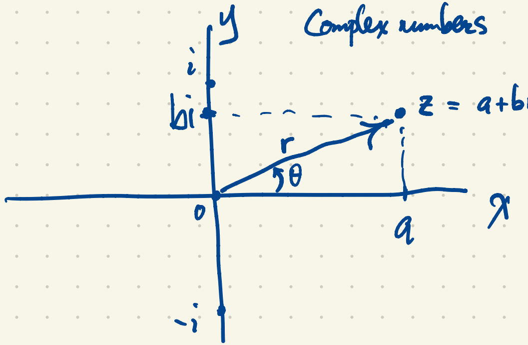
What are the automorphisms of $E = \mathbb{Q}[\alpha]$, $\alpha = \sqrt[3]{2}$? If $\phi \in \operatorname{Aut} E$ then $\phi(\alpha)^3 = \phi(\alpha^3) = \phi(2) = 2$

In $\mathbb{C}$, every poly. $f(x) \in \mathbb{C}[x]$ of degree n factors as $f(x) = a(x-r_1)(x-r_2)\cdots(x-r_n)$ $(a, r_1, r_2, \ldots, r_n \in \mathbb{C})$.

eg. $x^n - 1 = (x-1)(x-\xi)(x-\xi^2)(x-\xi^3)\cdots(x-\xi^{n-1})$ where $\xi = e^{2\pi i/n}$.

de Moivre's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Complex numbers $\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$, $i = \sqrt{-1}$



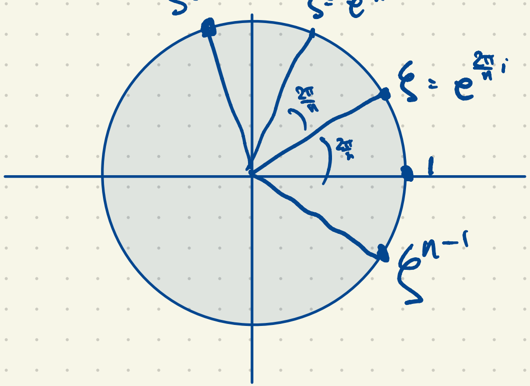
Every $z \in \mathbb{C}$ has unique representation as $z = a+bi$ $(a, b \in \mathbb{R})$ in rectangular coordinates

$a = \operatorname{Re} z$ = real part of z

$b = \operatorname{Im} z$ = imaginary part of z.

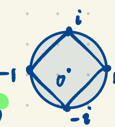
$r = |z| = \sqrt{a^2+b^2}$

The roots of $x^n - 1$ are the n^{th} roots of unity: $1, \xi, \xi^2, \ldots, \xi^{n-1}$ forming the vertices of a regular n-gon inscribed in the unit circle $|z| = 1$.



Eg. $n = 4$
The fourth roots of unity are $\pm 1, \pm i$

$1, e^{\frac{i\pi}{2}} = i, e^{\frac{2i\pi}{2}} = e^{i\pi} = -1, e^{\frac{3i\pi}{2}} = -i$



Euler's Formula $e^{i\pi} = -1$

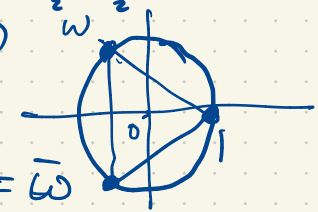
$e^{i\pi} + 1 = 0$

Eg. $n=3$: The three cube roots of unity in $\mathbb{C}$ are $1, \omega, \omega^2$ where

$\omega = e^{\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$

$\omega = \frac{-1 \pm \sqrt{-3}}{2}$



$\omega^2 = \overline{\omega}$

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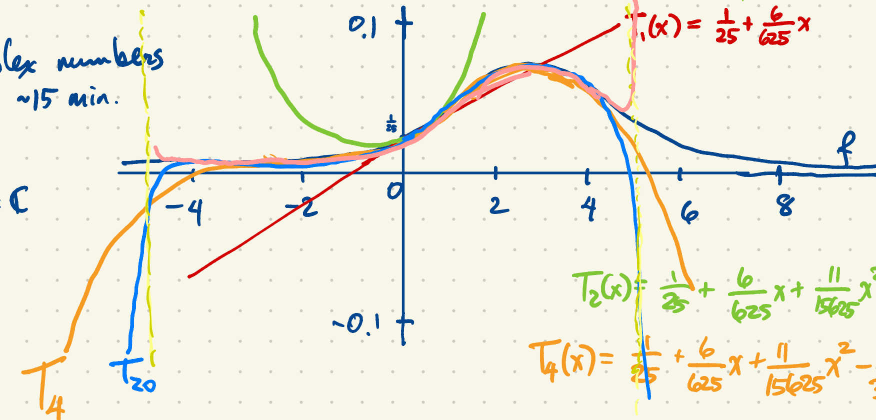
instructional videos $\longrightarrow$ complex numbers
~15 min.

Eg. consider $f(x) = \dfrac{1}{x^2-6x+25}$

This function has poles at $x = 3 \pm 4i \in \mathbb{C}$

with $|3 \pm 4i| = 5$



$T_1(x) = \frac{1}{25} + \frac{6}{625}x$

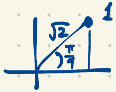
$T_2(x) = \frac{1}{25} + \frac{6}{625}x + \frac{11}{15625}x^2$

$T_4(x) = \frac{1}{25} + \frac{6}{625}x + \frac{11}{15625}x^2 - \frac{84}{390625}x^3 - \frac{779}{9765625}x^4$

By the Binomial Theorem

$$(1+i)^{11} = 1 + 11i - 55 - 165i + 330 + 462i - 462 - 330i + 165 + 55i - 11 - i = -32 + 32i$$

Much faster way to evaluate powers $z^n = (x+iy)^n = x^n + n x^{n-1} yi + \cdots + i^n y^n$ (Binomial Theorem)



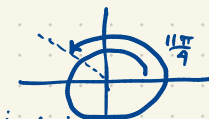
$1+i = \sqrt{2}e^{i\frac{\pi}{4}}$

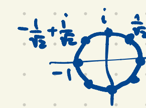
$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$(1+i)^{11} = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{11} = 32\sqrt{2}\, e^{\frac{i 11\pi}{4}}$$

$$= 32\sqrt{2}\cdot\left(\frac{-1+i}{\sqrt{2}}\right)$$

$$= -32 + 32i$$





$= e^{\frac{2\pi i}{8}} = e^{\frac{\pi i}{4}}$ eighth root of unity

$\zeta = \frac{1+i}{\sqrt{2}}$

$\zeta^{11} = \zeta^3 = \frac{-1+i}{\sqrt{2}}$

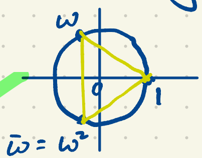
n^{th} roots of $z = re^{i\theta}$, $r = |z|$

all complex numbers whose n^{th} power is z

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}e^{\frac{i\theta}{n}},\ r^{\frac{1}{n}}e^{\frac{i\theta}{n}+\frac{2\pi i}{n}},\ r^{\frac{1}{n}}e^{\frac{i(\theta+4\pi)}{n}},\ \ldots,\ r^{\frac{1}{n}}e^{i\frac{\theta+2(n-1)\pi}{n}}$$

i.e. $r^{\frac{1}{n}}e^{i\frac{\theta+2k\pi}{n}}$, $k = 0, 1, 2, \ldots, n-1$

Cube roots of unity in $\mathbb{C}$: $1, \omega, \omega^2 = \bar{\omega}$



$$\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2} = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$$

$$\omega^2 + \omega + 1 = 0$$

ω is a root of $x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$

$\tau(\omega) = \omega^2$

$$\begin{array}{ccc} \mathbb{Q}[\omega] & & G = \langle \tau \rangle \\ 2 \Big| & \text{Galois correspondence} & 2 \Big| \\ \mathbb{Q} & & \langle \iota \rangle \end{array}$$

$G = \operatorname{Aut} \mathbb{Q}[\omega] = \langle \tau \rangle = \{\iota, \tau\}$
where $\tau(z) = \bar{z}$

Now let $\alpha = \sqrt[3]{2}$, $F = \mathbb{Q}[\alpha]$.

The min. poly. of α over $\mathbb{Q}$ is $x^3 - 2 \in \mathbb{Q}[x]$.

$F = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$, $\operatorname{Aut} F = \{\iota\}$

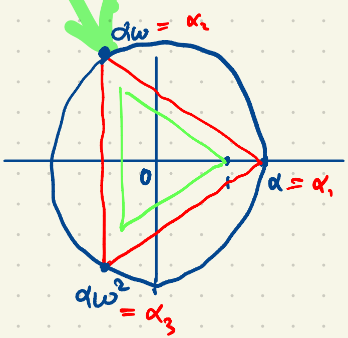
$$\begin{array}{c} F \\ 3 \Big| \\ \mathbb{Q} \end{array}$$

$x^3 - 2 = (x-\alpha)\underline{(x^2 + \alpha x + \alpha^2)}$

The other roots of $x^3 - 2$ are not in $F = \mathbb{Q}[\alpha]$ ie. the extension $F \supset \mathbb{Q}$ is not normal.

scale by factor of α

$x^3 - 2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ where $\alpha_1 = \alpha$, $\alpha_2 = \alpha\omega$, $\alpha_3 = \alpha\omega^2$



$$\alpha^3 = 2$$
$$(\alpha\omega)^3 = \alpha^3\omega^3 = 2 \cdot 1 = 2$$
$$(\alpha\omega^2)^3 = \alpha^3\omega^6 = 2 \cdot 1 = 2$$

$$x^3 - 2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$$
$$= (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$$

basis $\{1, \omega\}$ so $[E:F] = 2$

basis $1, \alpha, \alpha^2$ so $[F:\mathbb{Q}] = 3$

$$E \supset F \supset \mathbb{Q}$$

$E = \mathbb{Q}[\alpha_1, \alpha_2, \alpha_3] = \mathbb{Q}[\alpha, \omega]$, $F = \mathbb{Q}[\alpha]$

$[E:\mathbb{Q}] = 2 \cdot 3 = 6$

$\omega = \frac{1}{2}\alpha_1^2\alpha_2$
$= \frac{1}{2} \cdot 2^{2/3} \cdot 2^{1/3}\omega$
$= \omega$

There are $3! = 6$ permutations of $\alpha_1, \alpha_2, \alpha_3$.

x	$\sigma(x)$	$\tau(x)$ ← complex conjugation
α_1	α_2	α_1
α_2	α_3	α_3
α_3	α_1	α_2
α	$\alpha\omega$	α
ω	ω	ω^2

In $S_3 = \langle \sigma, \tau \rangle$, $\sigma = (1\,2\,3)$, $\tau = (2\,3)$.

$$\sigma(\omega) = \sigma\left(\frac{\alpha_2}{\alpha_1}\right) = \frac{\sigma(\alpha_2)}{\sigma(\alpha_1)} = \frac{\alpha_3}{\alpha_2} = \frac{\alpha\omega^2}{\alpha\omega} = \omega$$

$\sigma\left(\frac{\alpha\omega}{\alpha}\right)$

$$\tau(\omega) = \tau\left(\frac{\alpha_2}{\alpha_1}\right) = \frac{\alpha_3}{\alpha_1} = \frac{\alpha\omega^2}{\alpha} = \omega^2 = \bar{\omega}$$

$\sigma^2\tau = \tau\sigma$

$\sigma\tau = \tau\sigma^2$

Double lines indicate normality.

Using right-to-left composition

A subgroup $H \le G$ is normal if its left and right cosets agree i.e. $gH = Hg$ for all $g \in G$.

$\sigma^2\tau = (132)(23) = (13)$

$\sigma\tau = (123)(23) = (12)$

$\tau = (23)$

Eg. in $G = S_3$, $H = \langle\sigma\rangle = \langle(123)\rangle$ is normal.

eg. $(12)H = (12)\{\underset{id}{()},\ \underset{\sigma}{(123)},\ \underset{\sigma^2}{(132)}\} = \{(12), (23), (13)\}$

$(12)(123) = (1)(23) = (23)$

$H(12) = \{(), (123), (132)\}(12) = \{(12), (13), (23)\}$

E is the splitting field of

$x^3-2 = (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$

$E = \mathbb{Q}[\alpha,\alpha\omega,\alpha\omega^2] = \mathbb{Q}[\alpha,\omega]$

$\langle\tau\rangle$ is a subgroup of G which is not normal in G.

$(13)\langle\tau\rangle = (13)\{(), (23)\} = \{(13), (132)\}$

$\langle\tau\rangle(13) = \{(), (23)\}(13) = \{(13), (123)\}$

The extension $\underset{\mathbb{Q}[\alpha_1]}{\mathbb{Q}[\alpha]} \supset \mathbb{Q}$ of degree $[\mathbb{Q}[\alpha]:\mathbb{Q}] = 3$ is not normal

since the min. poly. of α over $\mathbb{Q}$ is x^3-2

with $\mathbb{Q}[\alpha]$ containing only one of the three roots of x^3-2.

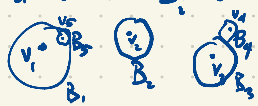
In $E=\mathbb{Q}[\alpha,\omega]$ the splitting field of $x^3-2=(x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$, can we find a single element $\beta\in E$ generating E i.e. $E=\mathbb{Q}[\beta]=\{a_0+a_1\beta+a_2\beta^2+\cdots+a_5\beta^5 : a_0,a_1,\dots,a_5\in\mathbb{Q}\}$?
Such an element β must be in E but not in $\mathbb{Q}[\omega]\cup\mathbb{Q}[\alpha]\cup\mathbb{Q}[\alpha\omega]\cup\mathbb{Q}[\alpha\omega^2]$.
In a 6-dimensional vector space, we must find a vector not contained in this union of four proper subspaces of dimension 2, 3, 3, 3 respectively.

In $\mathbb{R}^3$, can $\mathbb{R}^3$ be a union of finitely many proper subspaces? No because each proper subspace of $\mathbb{R}^3$ has only dimension ≤ 2 so it covers a slice of the unit ball of volume 0. A finite union of proper subspaces covers zero volume of the unit ball; it can never cover the total volume $\frac{4}{3}\pi$ of the unit ball.

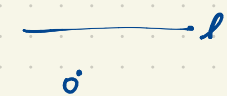
In $\mathbb{Q}^3$, i.e. points of $\mathbb{R}^3$ with rational coordinates, can $\mathbb{Q}^3=U_1\cup U_2\cup U_3\cup\cdots\cup U_k$ with $U_i\le\mathbb{Q}^3$ proper subspaces? The volume of $\mathbb{Q}^3$ (as a subset of $\mathbb{R}^3$) is zero.

$\mathbb{Q}^3=\{v_1,v_2,v_3,v_4,\dots\}$ is countably infinite.

Let $\varepsilon>0$. We will show that the volume of $\mathbb{Q}^3$ is at most ε.
Take a ball B_i of radius small enough centered at v_i such that its volume is less than $\frac{\varepsilon}{2^i}$. $(i=1,2,3,4,\dots)$



$\bigcup_{i=1}^{\infty}B_i$ has volume $<\frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{8}+\frac{\varepsilon}{16}+\cdots=\varepsilon$. Now $\mathbb{Q}^3\subset\bigcup_{i=1}^{\infty}B_i$, so $\mathrm{Vol}(\mathbb{Q}^3)<\varepsilon$.

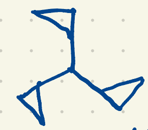


Try another approach. Suppose $\mathbb{Q}^3=U_1\cup U_2\cup\cdots\cup U_k$, $U_i\le\mathbb{Q}^3$ proper distinct subspaces, so $\dim U_i\in\{0,1,2\}$.
Take a line $\ell\subset\mathbb{Q}^3$ not through the origin. Then ℓ is contained in at most one of the subspaces U_i. With careful choice we may assume ℓ is not contained in any U_i. (Not hard.) Each U_i intersects ℓ in at most one point. This is a contradiction.

Galois theory handout: ignore "separable" for now.

Example of an extension $E \supset \mathbb{Q}$ of degree 3 with $G = \mathrm{Aut}\, E$ of order 3?

$f(x) = x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x]$ is irreducible



has exactly 3 symmetries

$f(x) = (x-\alpha)(x-\beta)(x-\gamma)$ where $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$

$$\alpha^3 = 1 + 2\alpha - \alpha^2$$
$$\alpha^4 = \alpha + 2\alpha^2 - \alpha^3 = \alpha + 2\alpha^2 - (1 + 2\alpha - \alpha^2) = -1 - \alpha + 3\alpha^2$$
$$\alpha^5 = 3 + 5\alpha - 4\alpha^2$$
$$\alpha^6 = -4 - 5\alpha + 9\alpha^2$$

Check that $\alpha^2 - 2$ is also a root of $f(x)$:

$f(\alpha^2-2) = (\alpha^2-2)^3 + (\alpha^2-2)^2 - 2(\alpha^2-2) - 1 = 0$ after collecting terms, so $\alpha^2 - 2 \in \{\alpha, \beta, \gamma\}$.

Can $\alpha^2 - 2 = \alpha$? No. If α is a root of $f(x) = x^3 + x^2 - 2x - 1$ and a root of $g(x) = x^2 - x - 2$ then α is a root of

$$\gcd(f(x), g(x)) = r(x) f(x) + s(x) g(x)$$ by Euclid's Algorithm

which is a factor of $f(x)$ of degree less than 3, a contradiction.

WLOG $\beta = \alpha^2 - 2$. Now $\beta^2 - 2$ is also a root of $f(x)$ by the same reasoning, so $\beta^2 - 2 \in \{\alpha, \beta, \gamma\}$.

As before, $\beta^2 - 2 \neq \beta$. If $\beta^2 - 2 = \alpha$ then $(\alpha^2-2)^2 - 2 = \alpha$: $\alpha^4 - 4\alpha^2 + 4 - 2 = \alpha$

$$\alpha^4 - 4\alpha^2 - \alpha + 2 = 0$$

but $\cancel{\gcd}(x^3 + x^2 - 2x - 1, \cancel{x^4 - 4x^2} - x + 2) = 1$, contradiction.

Better: $-1 - \alpha + 3\alpha^2 - 4\alpha^2 - \alpha + 2 = 0$

$1 - 2\alpha - \alpha^2 = 0$

So $\beta^2 - 2 = \gamma$. Now $\gamma^2 - 2 = \alpha$. Indeed

$$\gamma^2 - 2 = (\beta^2-2)^2 - 2 = ((\alpha^2-2)^2 - 2)^2 - 2 = \alpha.$$

The map $x \mapsto x^2 - 2$ gives a cyclic permutation $\sigma: \alpha \mapsto \beta \mapsto \gamma \mapsto \alpha$.

The field $E = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$ of degree $[E:\mathbb{Q}] = 3$ (a cyclic cubic extension)

has automorphism group $G = \mathrm{Aut}\, E = \langle \sigma \rangle = \{\iota, \sigma, \sigma^2\}$, cyclic of order 3.

$$\begin{tikzcd}
E = \mathbb{Q}[\alpha] \arrow[d, dash, "3"'] \arrow[rrd, leftrightarrow, red] & & G = \langle \sigma \rangle \arrow[d, dash, "3"] \arrow[lld, leftrightarrow, red] \\
\mathbb{Q} & & \langle 1 \rangle
\end{tikzcd}$$

Galois Correspondence

Exercise: Find three 3×3 matrices over $\mathbb{Q}$ A, B, C which are roots of $f(x)$ satisfying

$B = A^2 - 2I$ $\qquad A^3 + A^2 - 2A - I = 0$

$C = B^2 - 2I$ $\qquad B^3 + B^2 - 2B - I = 0$

(not HW) $\quad A = C^2 - 2I$ $\qquad C^3 + C^2 - 2C - I = 0.$

(don't worry about this technical condition for now)

In a finite (separable) extension $E \supseteq \mathbb{Q}$ of degree $[E:\mathbb{Q}] = n$, there exists $\beta \in E$ such that $E = \mathbb{Q}[\beta]$.

degree $n < \infty$

(Theorem of the simple element or simple extension)

i.e. extension generated by a single element

Note: $\mathbb{C} \supset \mathbb{R}$ is a simple extension, $\mathbb{C} = \mathbb{R}[i]$.

$\mathbb{R} \supset \mathbb{Q}$ is not a simple extension. There is no $\beta \in \mathbb{R}$ satisfying $\mathbb{Q}[\beta] = \mathbb{R}$.

or $\mathbb{Q}(\beta) = \mathbb{R}$

NOVEMBER 2024

SUN	MON	TUE	WED	THU	FRI	SAT
27	28	29	30	31	1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

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Mon Nov 4
Test

Given a number field $E \supseteq \mathbb{Q}$ of degree $n = [E:\mathbb{Q}] < \infty$, there exists $\beta \in E$ such that $E = \mathbb{Q}[\beta]$ (Theorem of the Primitive Element: $E \supseteq \mathbb{Q}$ is a simple extension). It follows that $|\mathrm{Aut}\, E| \le n$. Why? $\underbrace{1, \beta, \beta^2, \beta^3, \cdots, \beta^n}_{n+1 \text{ elements in } E \supseteq \mathbb{Q}}$ are linearly dependent so $a_0 + a_1\beta + a_2\beta^2 + \cdots + a_n\beta^n = 0$ for some $a_0, a_1, \dots, a_n \in \mathbb{Q}$, not all zero. Actually $a_n \neq 0$, otherwise $1, \beta, \beta^2, \cdots, \beta^{n-2}$ would generate the extension, a contradiction. After dividing by $a_n \neq 0$ we get

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n \in \mathbb{Q}[x]$$

as the minimal polynomial of β. Over $\mathbb{C}$ there exist $\beta_1, \beta_2, \cdots, \beta_n \in \mathbb{C}$ such that $\beta_1 = \beta$.

$$f(x) = (x-\beta_1)(x-\beta_2)\cdots(x-\beta_n)$$

If $\sigma \in \mathrm{Aut}\, E$ then σ must permute the n roots $\beta_1, \cdots, \beta_n$. (but $\beta_1, \cdots, \beta_n$ are not necessarily in $E = \mathbb{Q}[\beta]$.) Think of $f(x) = x^3 - 2$, $\beta = \sqrt[3]{2}$.

$$\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0 = 0$$
$$\Rightarrow \sigma(\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0) = 0$$
$$\sigma(\beta)^n + a_{n-1}\sigma(\beta)^{n-1} + \cdots + a_1\sigma(\beta) + a_0 = 0$$
$$\Rightarrow \sigma(\beta) \text{ is a root of } f(x).$$

If $\beta_1 = \beta, \beta_2, \cdots, \beta_r \in E$ and $\beta_{r+1}, \cdots, \beta_n \notin E$ then there exist automorphisms mapping $\beta = \beta_1$ to any of $\beta_1, \cdots, \beta_r$. Behind this fact is the explanation coming from the First Isomorphism for Ring Theory:

The evaluation map
$$\mathbb{Q}[x] \longrightarrow \mathbb{Q}[\beta]$$
$$g(x) \longmapsto g(\beta)$$
is a homomorphism of rings.

This map is onto, by definition, but not one-to-one. The kernel of this homomorphism is the principal ideal $(f(x)) = \{u(x)f(x) : u(x) \in \mathbb{Q}[x]\}$. So $\mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}[\beta] = E$

But in the same way we can evaluate at any of the $\beta_1, \dots, \beta_r \in E$ to get $\mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}[\beta_i]$ $(1 \le i \le r)$

$$
\begin{tikzcd}
 & \mathbb{Q}[x]/(f(x)) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\mathbb{Q}[\beta_1] \arrow[rr, "\cong"] \arrow[d, equal] & & \mathbb{Q}[\beta_i] \arrow[d, equal] \\
E & & E
\end{tikzcd}
\qquad (1 \le i \le r)
$$

This gives r isomorphisms $E \to E$. $|\mathrm{Aut}(E)| = r \in \{1, 2, \dots, n\}$
where r is how many of the roots of $f(x)$ lie in $E = \mathbb{Q}[\beta]$.

When $r = n$ (all roots of $f(x)$ lie in E) then the extension $E \supseteq \mathbb{Q}$ is a Galois extension and we have a one-to-one Galois correspondence between subfields of E and subgroups of $G = \mathrm{Aut}\, E$.

Wait: what if $f(x)$ has repeated roots? Is it possible for an irreducible polynomial $f(x)$ to have repeated roots?

A field F is separable if every irreducible poly $f(x) \in F[x]$ has only simple roots (no multiple roots) in any extension.

Theorem $\mathbb{Q}$ is separable.

Proof Let $f(x) \in \mathbb{Q}[x]$ be irreducible of degree $n \geq 2$. If $f(x)$ has a repeated root $\alpha \in \mathbb{C}$ then
$f(x) = (x-\alpha)^2 g(x)$, $g(x) \in \mathbb{C}[x]$ of degree $\geq n-2$. Continue on Monday ...