Math 4520

Fields

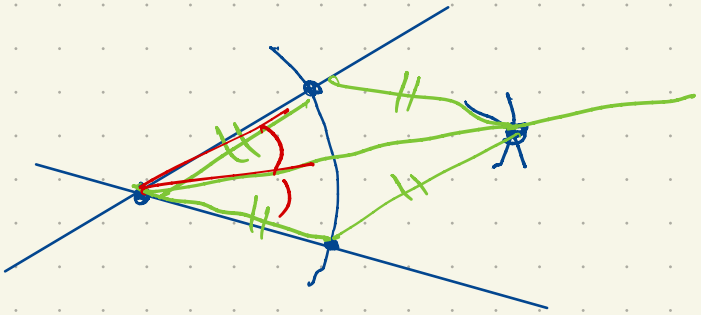
Book 3

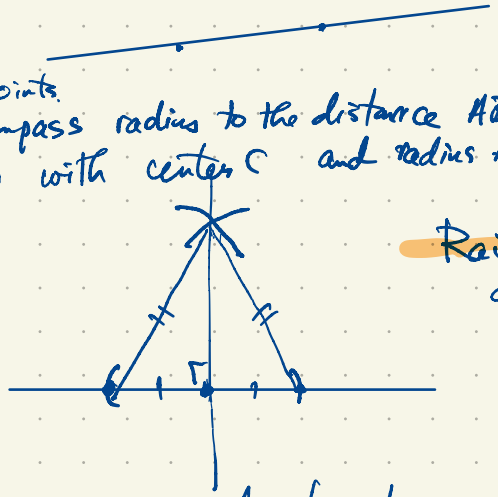
Suppose $\theta \in \mathbb{C}$ is a root of some nonzero poly $m(x) \in \mathbb{Q}[x]$.
Then take the smallest degree such $m(x)$. Assuming $m(x)$ is monic, this is the minimal polynomial of θ (over $\mathbb{Q}$). It is irreducible in $\mathbb{Q}[x]$.
Conversely, if $m(x) \in \mathbb{Q}[x]$ is a monic irreducible poly. having θ as a root, then it is the minimal poly. of θ. (over $\mathbb{Q}$).
To find $m(x)$: look at $1, \theta, \theta^2, \theta^3, \dots$ until you find a power θ^n which is a linear combination of smaller powers $1, \theta, \theta^2, \dots, \theta^{n-1}$.
$\theta^n = a_0 \cdot 1 + a_1\theta + a_2\theta^2 + \dots + a_{n-1}\theta^{n-1}$ $\iff$ θ is a root of $x^n - a_{n-1}x^{n-1} - \dots - a_1 x - a_0 \in \mathbb{Q}[x]$ (monic).

Straightedge and Compass Constructions

Here we allow certain operations:
Given two marked points on a page, we can mark the unique line through those two points.
Given any pair of points A, B we can set the compass radius to the distance AB and given a further point C, we can then mark the circle (or any arc thereof) with center C and radius AB.

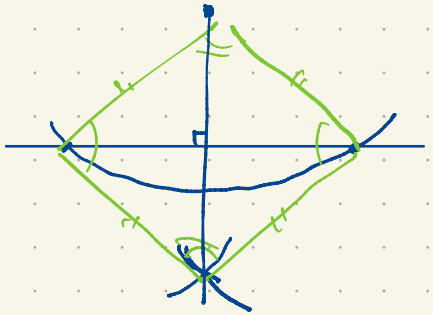
Using these tools, we can bisect any angle:



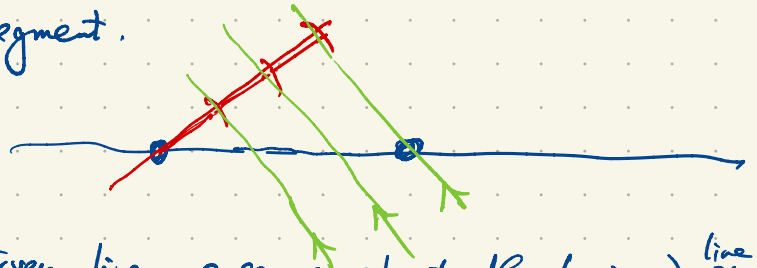


Raise a perpendicular to any line through any given point on that line

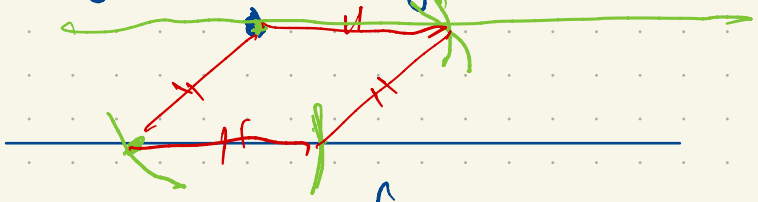
Drop a perpendicular to a line from any point not on the line:

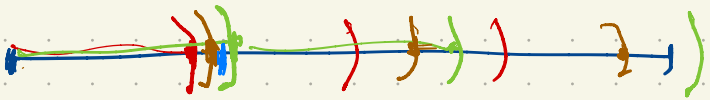


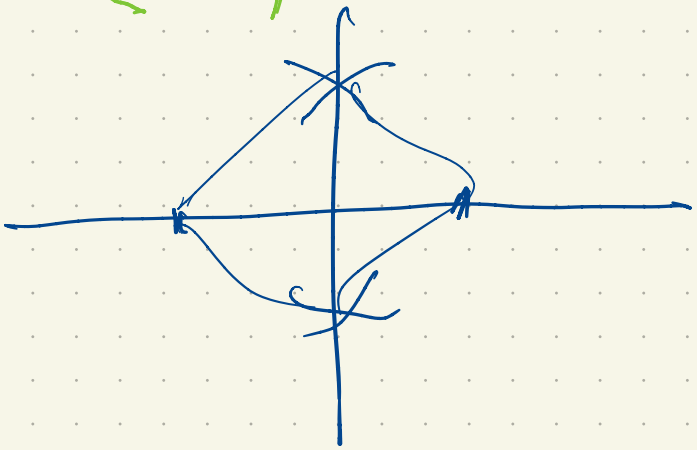
We can trisect any line segment.

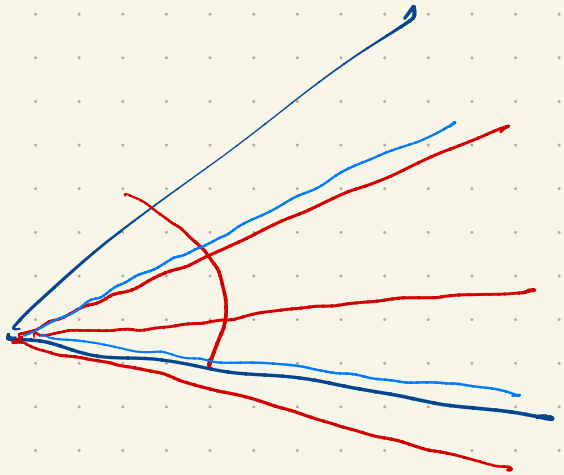


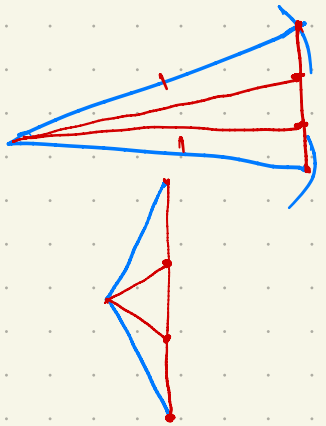
Through any point not on a given line, we can construct the (unique) line parallel to the given line.



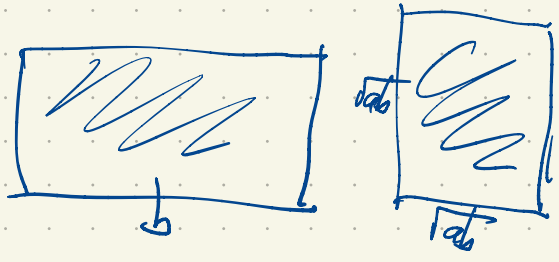




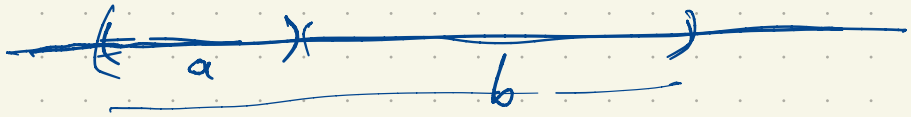


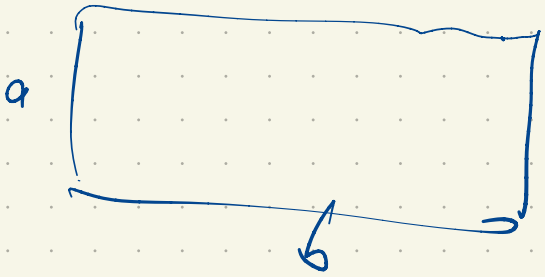


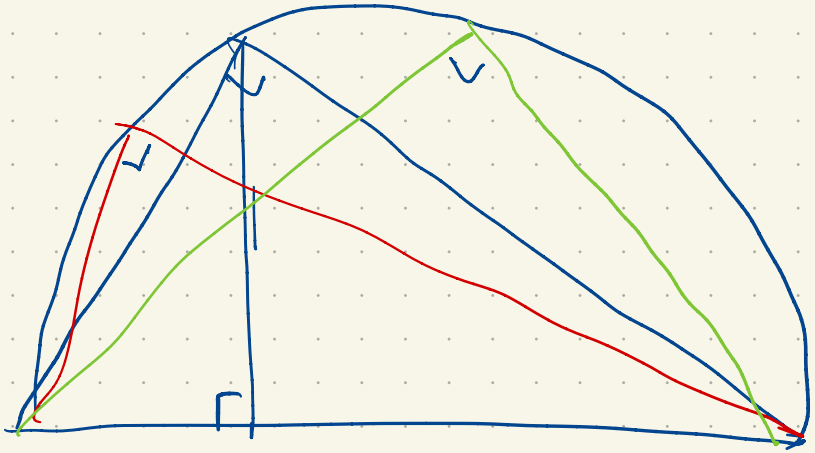
Given line segments of length a and b, we can construct a line segment of length $\sqrt{ab}$ (the geometric mean of a and b).

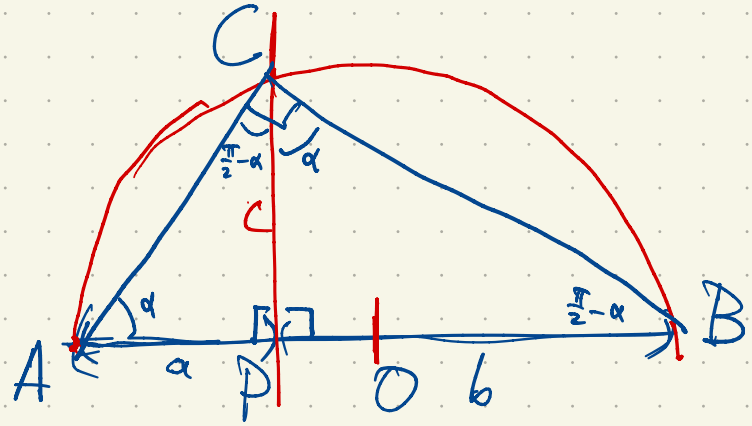


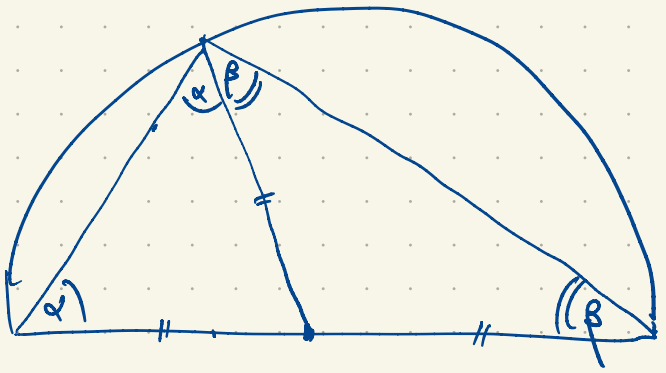
Given a and b,











$\alpha + (\alpha+\beta) + \beta = \pi$

$\alpha+\beta = \frac{\pi}{2}$

Triangle similar

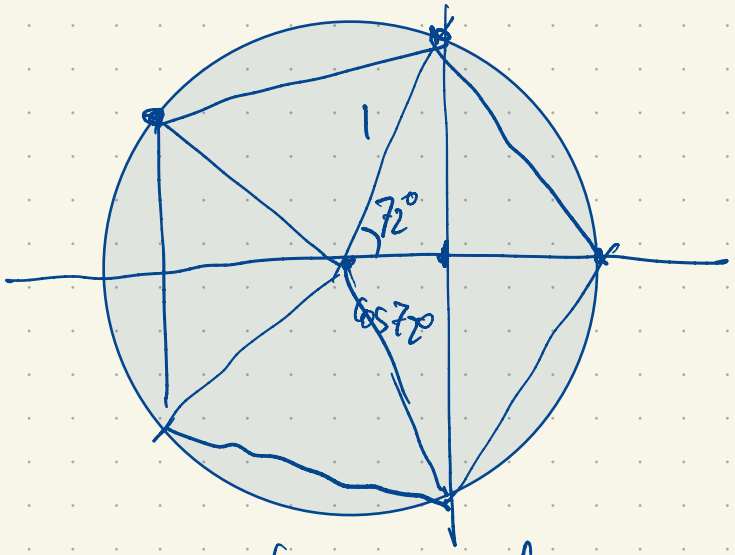
$\triangle APC \sim \triangle CPB$

by comparing corresponding angles $\alpha, \frac{\pi}{2}, \frac{\pi}{2}-\alpha$

$\frac{AP}{PC} = \frac{PC}{PB}$ i.e. $\frac{a}{c} = \frac{c}{b} \Rightarrow c^2 = ab$

$\Rightarrow c = \sqrt{ab}$.

Using our basic constructions (above), we can construct a regular pentagon.



circle of radius 1.

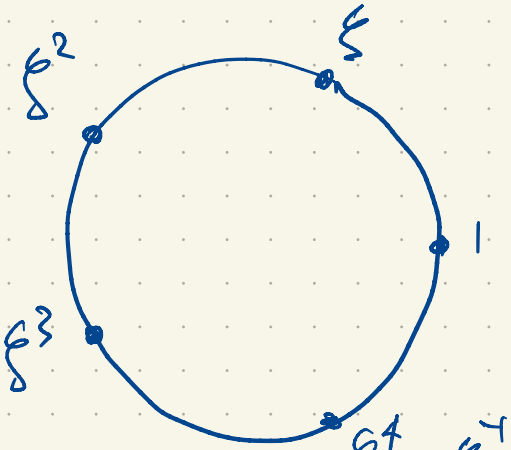


Construct a line segment of length $\cos 72° = \cos\frac{2\pi}{5}$.



$\zeta = e^{2\pi i/5}$, $\zeta^5 = 1$.

$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$



$\zeta^4 = \zeta^{-1} = \bar{\zeta} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

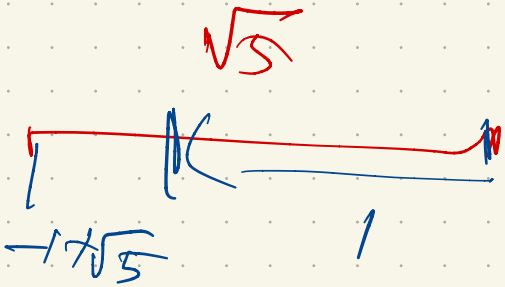
$2\cos\frac{2\pi}{5} = \zeta + \zeta^{-1} = \alpha$ in $\mathbb{Q}[\zeta] \supset \mathbb{Q}$ extension of degree 4

ζ is a root of $x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$

so ζ is a root of $x^4+x^3+x^2+x+1$ which is irreducible; it is the min. poly. of ζ, i.e. $\zeta^4+\zeta^3+\zeta^2+\zeta+1 = 0$.

$\alpha = \zeta + \zeta^{-1}$

$\alpha^2 = \zeta^2 + \zeta^{-2} + 2$



$\alpha + \alpha^2 = 2 + \zeta + \zeta^2 + \zeta^4 + \zeta^3$

$= 1 + (1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4)$

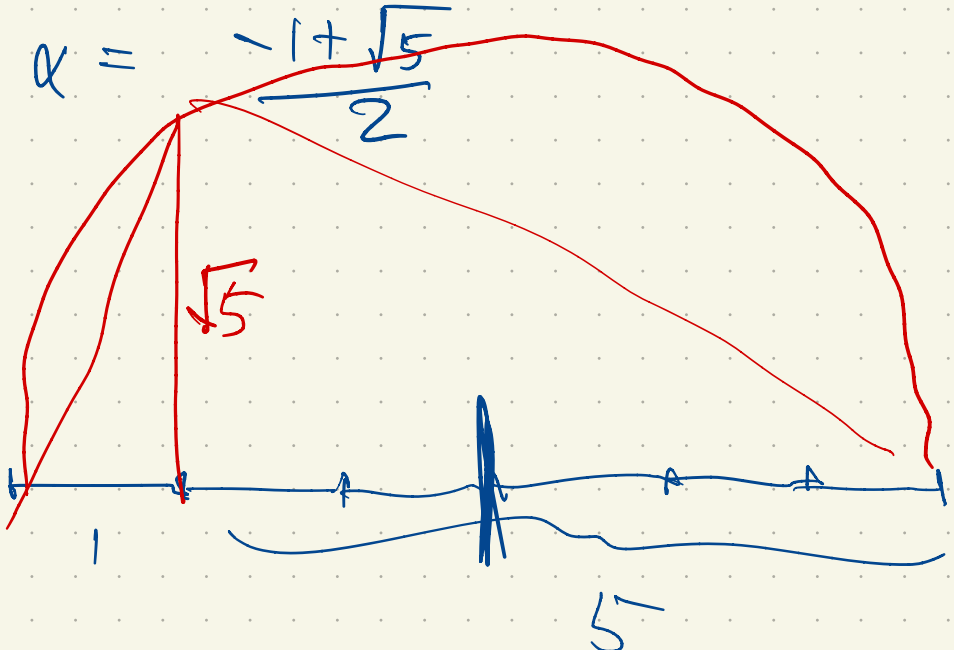
0

$\alpha^2 + \alpha - 1 = 0$

$\alpha = \dfrac{-1 \pm \sqrt{5}}{2}$

but $\alpha = 2\cos\dfrac{2\pi}{5} > 0$

so $\alpha = \dfrac{-1+\sqrt{5}}{2}$



For which n can we construct a regular n-gon? (using only straightedge and compass).

We know how to construct a regular n-gon for $n = 2^k \cdot 3^l \cdot 5^m \cdot 17^n \cdot 257^s \cdot 65537^t$ $\quad k \geq 0$, $l, m, n, s, t \in \{0, 1\}$

$$3 = 2+1$$
$$5 = 2^2+1$$
$$17 = 2^4+1$$
$$257 = 2^8+1$$
$$65537 = 2^{16}+1$$

A regular n-gon is constructible using straightedge and compass iff n is a power of 2 times a product of distinct Fermat primes. A Fermat prime is a prime that is 1 more than a power of 2.

The only Fermat primes that are known are 3, 5, 17, 257, 65537.

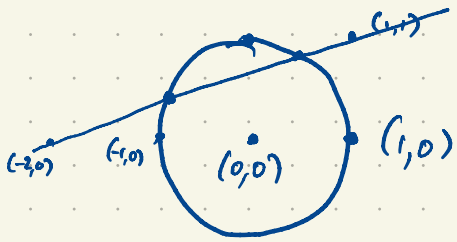
We'll show that a regular 9-gon cannot be constructed using straightedge and compass.
a $120°$ cannot be trisected.

A straightedge and compass construction is a sequence of steps, each of which is one of the following operations:

- join two previously marked points A, B by the line AB
- given previously constructed points A, B, C, construct the circle centered at C with radius AB
- mark the point of intersection of two previously constructed lines
- " " points " " " previously constructed line and circle, circles and circle.
- ...

We can assume $(0,0)$ is given and $(1,0)$ are given for free.

(In fact, we can see how to immediately construct any point $(a,b) \in \mathbb{Q}^2$.



We can construct the points of intersection of the circle $x^2+y^2=1$ with the line $y=\frac{1}{3}(x+2)$:

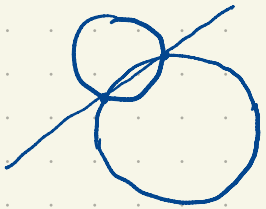
$$x^2+\left(\tfrac{1}{3}(x+2)\right)^2=1$$
$$x^2+\tfrac{1}{9}(x^2+4x+4)=1$$
$$9x^2+x^2+4x+4=9$$
$$10x^2+4x-5=0$$
$$x=\frac{-4\pm\sqrt{16+200}}{20}=\frac{-4\pm\sqrt{216}}{20}=\frac{-4\pm 6\sqrt{6}}{20}=\frac{-3\pm 3\sqrt{6}}{10}$$
$$\in \mathbb{Q}[\sqrt{6}].$$

We started with points in $F_0=\mathbb{Q}$.
After n steps, all points marked have coordinates in F_n.

$F_0\subseteq F_1\subseteq F_2\subseteq F_3\subseteq\cdots$

At step n, if the last operation was to intersect two lines, then $F_n=F_{n-1}$.
- - - - intersect a line with a circle, $F_n\geq F_{n-1}$ with $[F_n:F_{n-1}]=1$ or 2.
— — — intersect two circles

Intersect $\begin{cases}(x-a_1)^2+(y-b_1)^2=r_1^2\\(x-a_2)^2+(y-b_2)^2=r_2^2\end{cases}$



$a_i, b_i, r_i\in F_{n-1}$

$$\Leftrightarrow\begin{cases}x^2-2a_1x+a_1^2+y^2-2b_1y+b_1^2=r_1^2\\x^2-2a_2x+a_2^2+y^2-2b_2y+b_2^2=r_2^2\end{cases}$$
$$\Leftrightarrow\begin{cases}x^2-2a_1x+a_1^2+y^2-2b_1y+b_1^2=r_1^2\\2(a_2-a_1)x+(a_1^2-a_2^2)+2(b_2-b_1)y+(b_1^2-b_2^2)=r_2^2-r_1^2\end{cases}$$

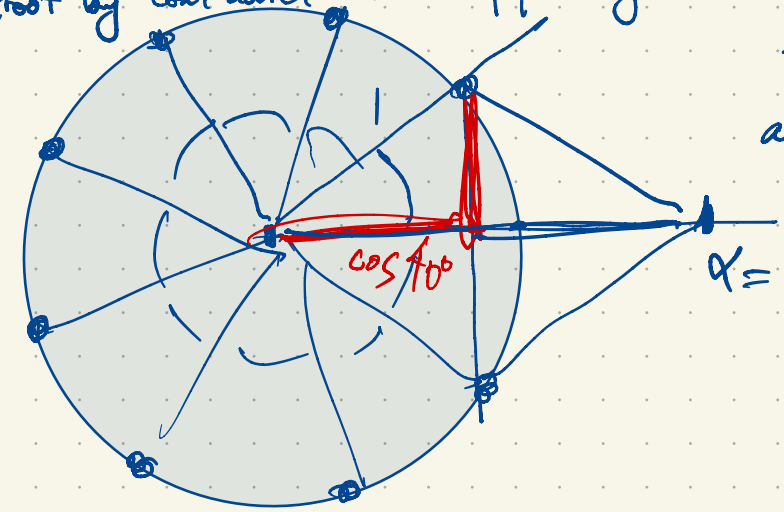
Gives at most two points of intersection (x,y) and they have coordinates in $F_n\geq F_{n-1}$, $[F_n:F_{n-1}]=1$ or 2.

$$F_n\geq F_{n-1}\geq\cdots\geq F_2\geq F_1\geq F_0=\mathbb{Q}$$
$$[F_n:\mathbb{Q}]=[F_n:F_{n-1}][F_{n-1}:F_{n-2}]\cdots[F_2:F_1][F_1:F_0] \quad \text{where each } [F_i:F_{i-1}]=1 \text{ or } 2$$
$$=2^k \text{ for some } k\in\{0,1,2,\ldots,n\}.$$

We'll show: you cannot construct a 40° angle using straightedge and compass.

So ... ... trisect a 120° angle

Proof by contradiction. Suppose you can construct a 40° angle. (equivalently, a regular nonagon 9-gon)



Then we can thereby construct a line segment of length $\cos 40°$

also we construct $\alpha = 2\cos 40° =$

$$\alpha = 2\cos\frac{2\pi}{9} = 2\cos 40°.$$

We'll show α is algebraic of degree 3:

its min. poly. is cubic so $[\mathbb{Q}[\alpha]:\mathbb{Q}] = 3$.

$\alpha \in F_n$, $[F_n:\mathbb{Q}] = 2^k$ (see previous argument)

$$F_n \supseteq \mathbb{Q}[\alpha] \supseteq \mathbb{Q}$$

$$[F_n:\mathbb{Q}] = 2^k = [F_n:\mathbb{Q}[\alpha]]\,[\mathbb{Q}[\alpha]:\mathbb{Q}]$$

$$= \text{integer} \cdot 3$$