Fields

Book I

Fields

Let F be a set containing distinct elements called 0 and 1 (thus $0 \neq 1$). Suppose addition, subtraction, multiplication and division are defined for all elements of F (except division by 0 is not defined).
Thus $a+b,\ a-b,\ ab,\ \frac{a}{d} \in F$ whenever $a, b, d \in F$ and $d \neq 0$.
Define $-a = 0 - a$.

If the following properties are satisfied by *all* elements $a, b, c, d \in F$ with $d \neq 0$, then F is a field.

$$a + b = b + a \qquad a + (b + c) = (a + b) + c \qquad ab = ba$$

$$a + 0 = a \qquad a(bc) = (ab)c \qquad 1a = a$$

$$a + (-a) = 0 \qquad a(b + c) = ab + ac \qquad \frac{a}{d}d = a$$

$$a + (-b) = a - b$$

$\mathbb{Q}^{2\times 2} = \{2\times 2 \text{ matrices over } \mathbb{Q}\} = \left\{\begin{bmatrix}a&b\\c&d\end{bmatrix} : a,b,c,d\in\mathbb{Q}\right\}$ is not a field.

$0 = \begin{bmatrix}0&0\\0&0\end{bmatrix}$, $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ identity

$A+0=A$, $AI = A = IA$

$\begin{bmatrix}1&0\\0&0\end{bmatrix}$ has no inverse. $A\begin{bmatrix}1&0\\0&0\end{bmatrix} = I$ has no solution for A.

Moreover, $AB \neq BA$ in general.

$\mathbb{Q}^{2\times 2}$ is a (non-commutative) ring with identity.

It has a subring $D = \left\{\begin{bmatrix}a&0\\0&d\end{bmatrix} : a,d\in\mathbb{Q}\right\}$ is a commutative subring with identity.

But D is not a field since it has non-invertible elements.

D has zero divisors: $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$. A field can never have zero divisors.

(If d is a zero divisor then $cd=0$ where $c,d\neq 0$ so $\left(\frac{c}{d}\right)d = c \neq 0$, contradiction)

$0\cdot\frac{1}{d} = \frac{0}{d} = \frac{cd}{d} =$

For a commutative ring R with identity, being able to divide is stronger than having no zero divisors.

An example of a commutative ring with identity having no zero divisors but not a field (division fails in general) is $\mathbb{Z}$

$\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix} = \begin{bmatrix}\frac{d}{ad-bc}&\frac{-b}{ad-bc}\\\frac{-c}{ad-bc}&\frac{a}{ad-bc}\end{bmatrix}$

Eg. $F = \left\{\begin{bmatrix}a&b\\2b&a\end{bmatrix} : a,b\in\mathbb{Q}\right\} \subset \mathbb{Q}^{2\times 2}$ is a subring, containing $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$.

If $\begin{bmatrix}a&b\\2b&a\end{bmatrix} \neq \begin{bmatrix}0&0\\0&0\end{bmatrix}$ then $\begin{bmatrix}a&b\\2b&a\end{bmatrix}^{-1} = \frac{1}{a^2-2b^2}\begin{bmatrix}a&-b\\-2b&a\end{bmatrix}$ (Note: $a^2-2b^2\neq 0$ since $\sqrt{2}\notin\mathbb{Q}$)

Why is F a commutative subring? Elements of F have the form

$\begin{bmatrix}a&b\\2b&a\end{bmatrix} = aI + bS$ where $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $S = \begin{bmatrix}0&1\\2&0\end{bmatrix}$ so $F = \{aI+bS : a,b\in\mathbb{Q}\}$ is the span of $\{I,S\}$

in $\mathbb{Q}^{2\times 2}$ (F is a 2-dimensional subspace of $\mathbb{Q}^{2\times 2}$, a 4-dimensional vector space).

$$(aI+bS)(cI+dS) = acI + (ad+bc)S + bdS^2 = (cI+dS)(aI+bS) \quad , \quad S^2 = \begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}0&1\\2&0\end{bmatrix} = 2I$$
$$= (ac+2bd)I + (ad+bc)S$$

Compare: $K = \mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a,b\in\mathbb{Q}\}$. is a field.

$$(a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$$
$$(a+b\sqrt{2})(c+d\sqrt{2}) = ac + (ad+bc)\sqrt{2} + 2bd = (ac+2bd) + (ad+bc)\sqrt{2}$$

Note: $F \cong K$ (they are isomorphic)

An explicit isomorphism $\phi : K \to F$ is given by $\phi(a+b\sqrt{2}) = \begin{bmatrix}a&b\\2b&a\end{bmatrix} = aI+bS$.

ϕ is bijective

$$\phi(x+y) = \phi(x)+\phi(y)$$
$$\phi(xy) = \phi(x)\phi(y)$$

Similarly $\left\{\begin{bmatrix}a&b\\-b&a\end{bmatrix} : a,b\in\mathbb{R}\right\} \subset \mathbb{R}^{2\times 2}$ is a subring isomorphic to $\mathbb{C}$.

An isomorphism $\mathbb{C} \to \left\{\begin{bmatrix}a&b\\-b&a\end{bmatrix} : a,b\in\mathbb{R}\right\}$ is $a+bi \longmapsto \begin{bmatrix}a&b\\-b&a\end{bmatrix}$ $(a,b\in\mathbb{R})$.

$\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\}$

$\alpha = 5+3\sqrt{2}$, $\beta = 7-\sqrt{2}$

$\alpha+\beta = 12+2\sqrt{2}$

$\alpha-\beta = -2+4\sqrt{2}$

$\alpha\beta = (5+3\sqrt{2})(7-\sqrt{2}) = 35-5\sqrt{2}+21\sqrt{2}-6 = 29+16\sqrt{2}$

$\frac{\alpha}{\beta} = \frac{5+3\sqrt{2}}{7-\sqrt{2}} = \frac{5+3\sqrt{2}}{7-\sqrt{2}} \cdot \frac{7+\sqrt{2}}{7+\sqrt{2}} = \frac{35+5\sqrt{2}+21\sqrt{2}+6}{47} = \frac{41+26\sqrt{2}}{47} = \frac{41}{47} + \frac{26}{47}\sqrt{2}$

Alternatively, $\frac{\alpha}{\beta} = \alpha\beta^{-1}$

in matrix representation: $\underbrace{\begin{bmatrix} 5 & 3 \\ 6 & 5 \end{bmatrix}}_{\alpha} \cdot \frac{1}{47}\begin{bmatrix} 7 & 1 \\ 2 & 7 \end{bmatrix} = \frac{1}{47}\begin{bmatrix} 41 & 26 \\ 52 & 41 \end{bmatrix}$

$\beta \longmapsto \begin{bmatrix} 7 & -1 \\ -2 & 7 \end{bmatrix}$

$\beta^{-1} \longmapsto \frac{1}{47}\begin{bmatrix} 7 & 1 \\ 2 & 7 \end{bmatrix}$

Similar: $\mathbb{Q}[\sqrt[3]{2}] = \mathbb{Q}[\theta]$, $\theta = \sqrt[3]{2}$.

$\{a+b\theta : a,b \in \mathbb{Q}\}$ is not a field, not even a ring since it's not closed under multiplication.

$\mathbb{Q}[\theta] = \{a+b\theta+c\theta^2 : a,b,c \in \mathbb{Q}\}$. <u>is</u> a field.

$\theta^3 = 2$

$\theta^4 = 2\theta$

$\theta^5 = 2\theta^2$

$\theta^6 = 4$

$\alpha = 5+3\theta$

$\beta = 7-\theta$

$\alpha+\beta = 12+2\theta$

$\alpha-\beta = -2+4\theta$

$\alpha\beta = (5+3\theta)(7-\theta) = 35-5\theta+21\theta-3\theta^2$

$= 35+16\theta-3\theta^2$

$$\frac{\alpha}{\beta} = \frac{5+3\theta}{7-\theta} = \boxed{a} + \boxed{b}\,\theta + \boxed{c}\,\theta^2 = \frac{251}{341} + \frac{182}{341}\theta + \frac{26}{341}\theta^2 = \frac{1}{341}\left(251 + 182\theta + 26\theta^2\right)$$

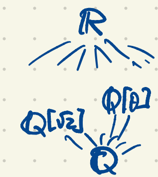
rational coefficients $a, b, c \in \mathbb{Q}$

$\theta = \sqrt[3]{2}$

$\theta^3 = 2$

$\theta^3 - 2 =$

θ is a root of $x^3 - 2 = (x-\theta)(x^2 + \theta x + \theta^2)$



$$5 + 3\theta = (a + b\theta + c\theta^2)(7 - \theta)$$
$$= 7a + (7b-a)\theta + (7c-b)\theta^2 - 2c$$
$$= (7a - 2c) + (7b - a)\theta + (7c - b)\theta^2$$

Hopefully

$$\begin{aligned} 7a - 2c &= 5 \\ -a + 7b &= 3 \\ -b + 7c &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 7 & 0 & -2 & 5 \\ -1 & 7 & 0 & 3 \\ 0 & -1 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 49 & -2 & 26 \\ -1 & 7 & 0 & 3 \\ 0 & -1 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -7 & 0 & -3 \\ 0 & 49 & -2 & 26 \\ 0 & 1 & -7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -7 & 0 & -3 \\ 0 & 1 & -7 & 0 \\ 0 & 49 & -2 & 26 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -49 & -3 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 341 & 26 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -49 & -3 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 1 & \frac{26}{341} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{251}{341} \\ 0 & 1 & 0 & \frac{182}{341} \\ 0 & 0 & 1 & \frac{26}{341} \end{array}\right]$$

$$\begin{array}{r} 26 \\ \times\ 7 \\ \hline 182 \end{array} \qquad \begin{array}{r} 49 \\ 26 \\ \hline 294 \\ 98 \\ \hline 1274 \end{array} \qquad \begin{array}{r} 341 \\ 3 \\ \hline 1023 \end{array}$$

$$-3 + 49 \cdot \frac{26}{341}$$
$$= -3 + \frac{1274}{341}$$
$$= \frac{-1023 + 1274}{341} = \frac{251}{341}$$

Check: $\frac{1}{341}(251 + 182\theta + 26\theta^2)(7-\theta) = \frac{1}{341}\left(1757 + 1023\theta + 0\theta^2 - 52\right)$

$$= \frac{1}{341}\left(1705 + 1023\theta\right)$$
$$= 5 + 3\theta \quad \checkmark$$

$\mathbb{Q}[\theta]$ is a cubic field extension of $\mathbb{Q}$: it is a 3-dimensional vector space over $\mathbb{Q}$, with basis $1, \theta, \theta^2$.

Alternatively, use 3x3 matrices to represent elements of $\mathbb{Q}[\theta]$.

Take $T = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ to represent θ. $T^3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I$

$E = \{ aI + bT + cT^2 : a,b,c \in \mathbb{Q} \} = \underbrace{\left\{ \begin{bmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{bmatrix} : a,b,c \in \mathbb{Q} \right\}}_{\text{This subring is a field.}} \subset \underbrace{\mathbb{Q}^{3\times 3}}_{\text{noncommutative ring with identity having zero divisors}}$

$\mathbb{Q}[\theta] \cong E$ via the isomorphism

$$\begin{array}{ccc} \mathbb{Q}[\theta] & & E \\ \cup\!\!\!\!\!| & & \cup\!\!\!\!\!| \\ a+b\theta+c\theta^2 & \mapsto & aI+bT+cT^2 \end{array}$$

Are there any fields "between" $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{2}]$, or between $\mathbb{Q}$ and $\mathbb{Q}[\theta]$?

Are there any fields "between" $\mathbb{R}$ and $\mathbb{C}$?

Suppose $\mathbb{R} \subset F \subset \mathbb{C}$ is a tower of fields (F is a subfield of $\mathbb{C}$ and $\mathbb{R}$ is a subfield of F).

$\subseteq$ vs $\subset$ '$\subset$' always means strict containment in this course.

$\leq$ vs $<$

Since $F \supset \mathbb{R}$, there exists $\alpha \in F$, $\alpha \notin \mathbb{R}$. Then $\alpha, 1$ are linearly independent over $\mathbb{R}$, i.e. $\alpha \neq a \cdot 1$ for any $a \in \mathbb{R}$. However $\mathbb{C}$ is 2-dimensional over $\mathbb{R}$ with basis $1, i$ (every complex number is uniquely expressible as $z = a\cdot 1 + b\cdot i$ with $a,b \in \mathbb{R}$). So $1, \alpha$ is a basis for F. So $F = \mathbb{C}$.

Is there any field extension $\mathbb{C} \subset F$ with F 2-dimensional over $\mathbb{C}$?
No, but there do exist fields $F \supset \mathbb{C}$ which are infinite-dimensional extensions.

Consider the ring $\mathbb{C}[x] = \{\text{polynomials in } x \text{ with complex coefficients}\}$
$= \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n : a_i \in \mathbb{C},\ n \geq 0\}$

This is a ring but not quite a field eg.

$$\frac{5 + 7x + ix^2}{3 - (4+i)x + 43x^2} \notin \mathbb{C}[x]$$

$\mathbb{C}(x)$ = field of fractions of $\mathbb{C}[x]$
= field of rational functions in x with complex coefficients

Just like constructing $\mathbb{Q}$ from $\mathbb{Z}$.

Another example of this: We'll construct a countably infinite subfield of $\mathbb{R}$ containing π.
This contains the subring $\mathbb{Q}[\pi] = \{a_0 + a_1\pi + a_2\pi^2 + \dots + a_n\pi^n : n \geq 0,\ a_i \in \mathbb{Q}\}$

$\pi \in \mathbb{Q}[\pi]$ has no (multiplicative) inverse in $\mathbb{Q}[\pi]$ since if

$$1 = \pi(a_0 + a_1\pi + a_2\pi^2 + \dots + a_n\pi^n) \qquad a_i \in \mathbb{Q},\ n \geq 0,$$

a contradiction since π is transcendental. (π would be a root of a nonzero polynomial $a_n x^{n+1} + a_{n-1}x^n + \dots + a_2 x^3 + a_1 x^2 + a_0 x - 1$)
(Lindemann 1800's)

$\mathbb{Q}(\pi) = \{\frac{a}{b} : a, b \in \mathbb{Q}[\pi],\ b \neq 0\}$ is the field of quotients of the ring $\mathbb{Q}[\pi]$

$\mathbb{Q}(\sqrt{2}) = \{\frac{a}{b} : a, b \in \mathbb{Q}[\sqrt{2}],\ b \neq 0\} = \mathbb{Q}[\sqrt{2}]$ is already a field. $\sqrt{2}$ is algebraic: it is a root of a nonzero poly. $x^2 - 2 \in \mathbb{Q}[x]$

Every $\alpha \in \mathbb{C}$ is either algebraic or transcendental, never both.

$$\underbrace{\mathbb{Q}}_{\text{countable}} \subset \underbrace{\mathbb{R}}_{\text{uncountable}} \subset \underbrace{\mathbb{C}}_{\text{uncountable}}$$

$$\underbrace{\mathbb{Q} \subset \mathbb{A}}_{\text{countable}} \subset \underbrace{\mathbb{C}}_{\text{uncountable.}}$$

$\mathbb{A} = \{\text{algebraic numbers}\}$.

$$\begin{array}{ccccc} & & \mathbb{A} & & \\ & \subset & & \subset & \\ \mathbb{Q} \subset \mathbb{A}\cap\mathbb{R} & & & & \mathbb{C} \\ & \subset & & \subset & \\ & & \mathbb{R} & & \end{array}$$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2$

$\mathbb{Q} = \{a_1, a_2, a_3, a_4, \dots\}$

$$\begin{array}{cccc} a_1+a_1x & a_1+a_2x & a_1+a_3x & \cdots \\ a_2+a_1x & a_2+a_2x & a_2+a_3x & \cdots \\ a_3+a_1x & a_3+a_2x & a_3+a_3x & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

$\mathbb{Q}[\pi]$ is a countably infinite ring
so $\mathbb{Q}(\pi)$ is a countably infinite field.

Elements of $\mathbb{Q}(\pi) \subset \mathbb{R}$ look like

$$\frac{53.8\pi^3 - 17\pi + \frac{53}{7}}{42\pi^2 + 119\pi + \frac{103}{648}}$$

Compare: $\mathbb{Q}(e) \subset \mathbb{R}$, another countable subfield of $\mathbb{R}$.

Actually $\mathbb{Q}(e) \cong \mathbb{Q}(\pi)$. An ~~isomorphism~~ isomorphism is $f(e) \longmapsto f(\pi)$ where $f(x) \in \mathbb{Q}(x)$.

$\cong \mathbb{Q}(x)$ (x being an indeterminate ie. an {abstract / general / generic} symbol)

$\mathbb{Q}(x) \longrightarrow \mathbb{Q}(\pi)$ evaluation

$\mathbb{Q}(x) \longrightarrow \mathbb{Q}(e)$

$\mathbb{Q}(x) \longrightarrow \mathbb{Q}(\sqrt{2})$ doesn't quite work eg. the image of $\frac{x^3+7x^2-3}{x^2-2} \in \mathbb{Q}(x)$ is undefined; you can't evaluate this at $\sqrt{2}$.

But the evaluation maps at $\pi, e, \sqrt{2}, \dots$

$$\left.\begin{array}{l} \mathbb{Q}[x] \longrightarrow \mathbb{Q}[\pi] \\ \mathbb{Q}[x] \longrightarrow \mathbb{Q}[e] \\ \mathbb{Q}[x] \longrightarrow \mathbb{Q}[\sqrt{2}] \\ \quad\vdots \end{array}\right\} \text{ all well-defined ring homomorphisms.}$$

If $\phi: R \to S$ where R, S are rings, we say ϕ is a <u>ring homomorphism</u> if

$$\left.\begin{array}{l} \phi(a+b) = \phi(a)+\phi(b) \\ \phi(ab) = \phi(a)\phi(b) \end{array}\right\} \text{ for all } a,b \in R$$

We don't necessarily require $\phi(1) = 1$; and in general the rings R, S may not have identity.

If R, S are rings with identity ($1_R \in R$, $1_S \in S$) we might consider only homomorphisms of rings with identity i.e. $\phi(1_R) = \phi(1_S)$.

* Suppose F, K are fields. If $\phi: F \to K$ is a ring homomorphism then either

(i) $\phi(F) = \{0\}$ i.e. $\phi(a) = 0$ for all $a \in F$, or (trivial)

(ii) ϕ is one-to-one i.e. $\phi(F) \subseteq K$ is a subfield isomorphic to F.

Any homomorphism $\mathbb{Q}(x) \to \mathbb{R}$ is either trivial or it has the form $\mathbb{Q}(x) \to \mathbb{Q}(a)$, $f(x) \mapsto f(a)$ is an evaluation at some transcendental number $a \in \mathbb{R}$.

We have ring homomorphisms $\mathbb{Q}[x] \to \mathbb{C}^{n\times n}$ ($n\times n$ complex matrices) where we evaluate at a matrix $A \in \mathbb{C}^{n\times n}$, i.e. $f(x) \mapsto f(A)$

$$\frac{47}{3}x^2 + \frac{18}{11}x - \frac{41}{7} \mapsto \frac{47}{3}A^2 + \frac{18}{11}A - \frac{41}{7}I$$

(*) In a field F, every ideal is either $\{0\}$ or F.

An <u>automorphism</u> of a field F is an isomorphism $\phi: F \to F$. Eg

(i) Automorphisms of $\mathbb{Q}[\sqrt{2}]$? We want $\phi: \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{2}]$ bijective with

$$\phi(a+b) = \phi(a)+\phi(b), \quad \phi(ab) = \phi(a)\phi(b).$$

- The identity $\phi(x) = x$ for all $x \in \mathbb{Q}[\sqrt{2}]$
- Conjugation $\phi(a+b\sqrt{2}) = a - b\sqrt{2}$ for all $a,b \in \mathbb{Q}$. (This is algebraic conjugation, not complex conjugation).

These are the only automorphisms of $\mathbb{Q}[\sqrt{2}]$.

If $\phi : F \to F$ is any automorphism of a field F then

$\phi(0) = \phi(0+0) = \phi(0) + \phi(0) \Rightarrow \phi(0) = 0$

$\phi(1) = \phi(1\cdot 1) = \phi(1)\cdot\phi(1)$ where $\underset{\phi(0)}{\phi(1) \neq 0}$ since ϕ is one-to-one. Multiply both sides by $\phi(1)^{-1}$ to get $\phi(1) = 1$.

$\phi(2) = \phi(1+1) = \phi(1) + \phi(1) = 1+1 = 2$

$\phi(3) = \phi(2+1) = \phi(2) + \phi(1) = 2+1 = 3$

So $3 + (-3) = 0$

$\underset{3}{\underset{\parallel}{\phi(3)}} + \underset{-3}{\phi(-3)} = \phi(0) = 0$

If $m,n \in \mathbb{Z}$ with $n \neq 0$,

$\phi(n\cdot\frac{m}{n}) = \phi(m) = m$

$\underset{n}{\underset{\parallel}{\phi(n)}}\,\phi(\frac{m}{n}) \Rightarrow \phi(\frac{m}{n}) = \frac{m}{n}$. So $\phi(x) = x$ for all $x \in \mathbb{Q}$.

$\phi(\sqrt{2})^2 = \phi(\sqrt{2}^2) = \phi(2) = 2 \Rightarrow \phi(\sqrt{2}) = \pm\sqrt{2}$.

If $\phi(\sqrt{2}) = \sqrt{2}$ then $\phi(a+b\sqrt{2}) = \phi(a) + \phi(b)\phi(\sqrt{2}) = a + b\sqrt{2}$ for all $a,b \in \mathbb{Q}$

If $\phi(\sqrt{2}) = -\sqrt{2}$ then $\phi(a+b\sqrt{2}) = \phi(a) + \phi(b)\phi(\sqrt{2}) = a + b(-\sqrt{2}) = a - b\sqrt{2}$.

If F is any field then $\operatorname{Aut} F = \{$all automorphisms of $F\}$ is a group under composition. Its identity is ι where $\iota : F \to F$, $\iota(x) = x$ for all $x \in F$ (the identity map).

$\operatorname{Aut}\mathbb{Q} = \{\iota\}$ is trivial.

$\operatorname{Aut}\mathbb{R} = \{\iota\}$ is trivial but why?

$\mathbb{Q}[\sqrt{2}] \subset \mathbb{R}$ has two automorphisms.

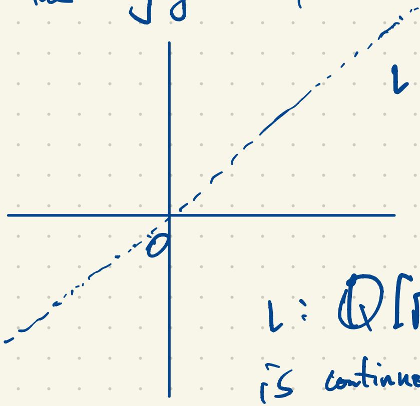
$\operatorname{Aut}\mathbb{Q}[\sqrt{2}]$ is a group of order 2.

$\operatorname{Aut}\mathbb{C}$ contains ι and τ = complex conjugation, $\tau(a+bi) = a-bi$ for all $a,b \in \mathbb{R}$.

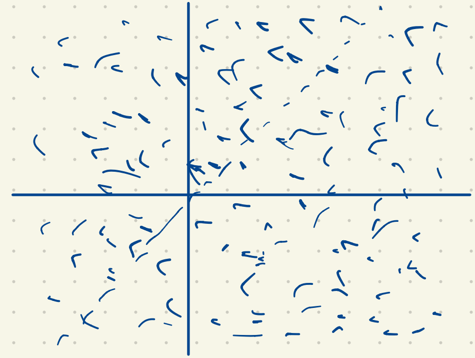
But $\operatorname{Aut}\mathbb{C}$ is uncountable. $\mathbb{C}$ has uncountably many automorphisms.

The only continuous automorphisms of $\mathbb{C}$ are ι, τ.

The conjugation $\phi \in \mathrm{Aut}\, \mathbb{Q}[\sqrt{2}]$ defined by $\phi(a+b\sqrt{2}) = a - b\sqrt{2}$ $(a, b \in \mathbb{Q})$ is badly discontinuous



$\iota : \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{2}]$
is continuous



graph of ϕ

ϕ is badly discontinuous.

$\mathbb{R}(x) = \{$rational functions of x with real coefficients$\}$ is a field.

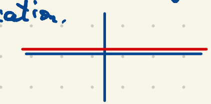
Can we replace "rational functions" with "functions" or "continuous functions" $\mathbb{R} \to \mathbb{R}$?

$\{$ functions $\mathbb{R} \to \mathbb{R} \}$

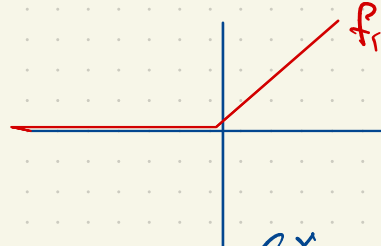
$\{$ continuous functions $\mathbb{R} \to \mathbb{R} \}$

are rings with zero divisors so they are not fields.

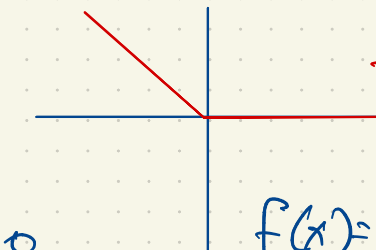
Commutative rings with identity under pointwise multiplication.



$f_1 f_2 = 0$



f_1



f_2

$$f_1(x) = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

$$f_2(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

$f_1 f_2 = 0$ but f_1, f_2 are nonzero functions.