Fields

Book II

Eg. $\alpha = \sqrt{2+\sqrt{2}}$

$\alpha^2 = 2+\sqrt{2}$

$\alpha^2 - 2 = \sqrt{2}$

$\alpha^4 - 4\alpha^2 + 4 = 2$

$\alpha^4 - 4\alpha^2 + 2 = 0$

The minimal poly. of α over $\mathbb{Q}$ is $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.

(Exercise: $f(x)$ is irreducible in $\mathbb{Q}[x]$ so it really is the min. poly. of α over $\mathbb{Q}$)

The roots of $f(x)$ are

$\alpha = \sqrt{2+\sqrt{2}}$

$-\alpha = -\sqrt{2+\sqrt{2}}$

$\beta = \sqrt{2-\sqrt{2}}$

$-\beta = -\sqrt{2-\sqrt{2}}$

$$f(x) = x^4 - 4x^2 + 2 = (x-\alpha)(x+\alpha)(x-\beta)(x+\beta)$$

In this case $E = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 + d\alpha^3 : a,b,c,d \in \mathbb{Q}\}$ contains all the roots of $f(x)$

so it is a normal extension of $\mathbb{Q}$. $\beta = (*) + (*)\alpha + (*)\alpha^2 + (*)\alpha^3 = \alpha^3 - 3\alpha$

$\alpha\beta = \sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}} = \sqrt{4-2} = \sqrt{2} = \alpha^2 - 2$

$\Rightarrow \beta = \dfrac{\alpha^2-2}{\alpha} \in \mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$

$\beta = \alpha - \dfrac{2}{\alpha} = \alpha - (4\alpha - \alpha^3) = \alpha^3 - 3\alpha$

$\alpha^4 - 4\alpha^2 + 2 = 0$

$\alpha^3 - 4\alpha + \dfrac{2}{\alpha} = 0 \Rightarrow \dfrac{2}{\alpha} = 4\alpha - \alpha^3$

$\alpha^4 = 4\alpha^2 - 2$

$\alpha^6 = 4\alpha^4 - 2\alpha^2$
$= 4(4\alpha^2 - 2) - 2\alpha^2$
$= 14\alpha^2 - 8$

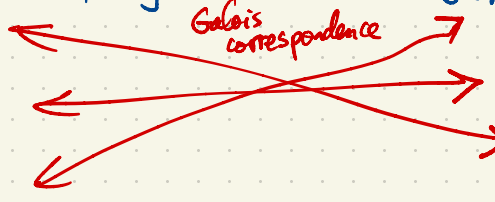Look for an automorphism $\sigma: E \to E$ $(E = \mathbb{Q}[\alpha])$ satisfying $\sigma(\alpha) = \beta$.

$\sigma(\beta) = \sigma(\alpha^3 - 3\alpha) = \sigma(\alpha)^3 - 3\sigma(\alpha) = \beta^3 - 3\beta = (\alpha^2-3\alpha)^3 - 3(\alpha^3 - 3\alpha) = (\alpha^3-3\alpha)((\alpha^3-3\alpha)^2 - 3)$

$= (\alpha^3-3\alpha)(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 3) = (\alpha^3-3\alpha)(14\alpha^2-8 - 6(4\alpha^2-2) + 9\alpha^2-3) = (\alpha^3-3\alpha)(-\alpha^2+1) = \alpha(\alpha^2-3)(-\alpha^2+1)$

$= \alpha(-\alpha^4+4\alpha^2-3) = \alpha(-(4\alpha^2-2)+4\alpha^2-3) = -\alpha$

$\sigma: \alpha \longmapsto \beta = \alpha^3 - 3\alpha \longmapsto -\alpha \longmapsto -\beta \longmapsto \alpha$

Aut $E = \langle\sigma\rangle$ of order 4 ; cyclic.

$\mathbb{Q}[\alpha]$
$2|$
$\mathbb{Q}[\sqrt{2}]$
$2|$
$\mathbb{Q}$

Galois correspondence

$G = $ Aut $E = \langle\sigma\rangle = \{1, \sigma, \sigma^2, \sigma^3\}$
$2|$
$\langle\sigma^2\rangle = \{1, \sigma^2\}$
$2|$
$\langle 1\rangle$

$\sigma^2(\sqrt{2}) = \sigma(\sigma(\sqrt{2}))$
$= \sigma(-\sqrt{2}) = \sqrt{2}$.

$\sigma(\sqrt{2}) = ?$
$\sqrt{2} = \alpha\beta$
$\sigma(\sqrt{2}) = \sigma(\alpha)\sigma(\beta)$
$= \beta\cdot(-\alpha)$
$= -\alpha\beta$
$= -\sqrt{2}$

$\sigma(\sqrt{2}) = \sigma(\alpha^2 - 2)$
$= \sigma(\alpha)^2 - 2 = \beta^2 - 2 = -\sqrt{2}$.

$\sigma(\alpha) = \beta$
$\sigma(-\alpha) = -\sigma(\alpha) = -\beta$
$\sigma(\beta) = -\alpha$
$\sigma(-\beta) = -\sigma(\beta) = \alpha$
$= \alpha$

Compare:

$E = \mathbb{Q}[\sqrt{2}]$ $G = \langle \tau \rangle$

$2 \vert$ $\vert 2$

$\mathbb{Q}$ $\langle 1 \rangle$

$G = \text{Aut } E = \{1, \tau\}$, $\tau(a + b\sqrt{2}) = a - b\sqrt{2}$

Degree 2 extension : quadratic extension

..   3   ..   :   cubic   ..

..   4   ..   :   quartic   ..

..   5   ..   :   quintic   ..

$\alpha = \sqrt[3]{2} = 2^{1/3}$

$E = \mathbb{Q}[\alpha] \supset \mathbb{Q}$ is an extension of degree

$[E : \mathbb{Q}] = 3$

with basis $1, \alpha, \alpha^2 = \sqrt[3]{4}$ $(\alpha^3 = 2)$
$\quad\quad\quad \underset{\sqrt[3]{2}}{\Vert}$

α has min. poly. $x^3 - 2 \in \mathbb{Q}[x]$ which is irreducible

In $\mathbb{R}[x]$, $f(x) = x^3 - 2 = (x - \alpha)(x^2 + \alpha x + \alpha^2)$

Subfields

$E = \mathbb{Q}[\alpha]$

$\vert 3$

$\mathbb{Q}$

If $E \supseteq F \supseteq \mathbb{Q}$ (i.e. F is an intermediate field) then the transitivity of degrees tells us $[E : \mathbb{Q}] = [E : F][F : \mathbb{Q}]$

$$\underline{\quad 3 \quad} \quad\quad \underset{\text{or } 1 \times 3}{3 \times 1}$$

If $[F : \mathbb{Q}] = 1$ then $\{1\}$ is a basis for F over $\mathbb{Q}$ so $F = \{a1 : a \in \mathbb{Q}\}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = \mathbb{Q}$

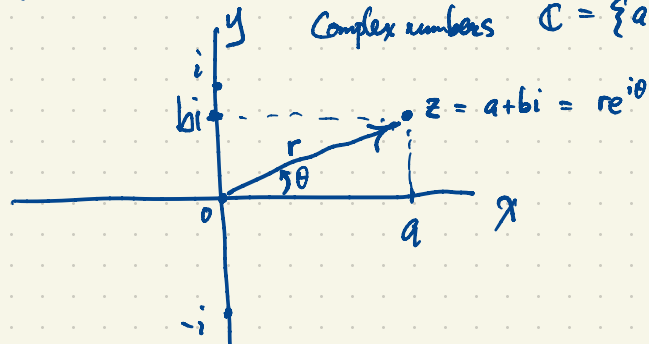If $[E : F] = 1$ then (similarly) $E = F$.

More generally if $E \supseteq F$ is an extension of prime degree $p = [E : F]$ then the only intermediate extensions are E and F.

What are the automorphisms of $E = \mathbb{Q}[\alpha]$, $\alpha = \sqrt[3]{2}$?   If $\phi \in \text{Aut } E$ then $\phi(\alpha)^3 = \phi(\alpha^3) = \phi(2) = 2$

In $\mathbb{C}$, every poly. $f(x) \in \mathbb{C}[x]$ of degree n factors as $f(x) = a(x-r_1)(x-r_2)\cdots(x-r_n)$ $(a, r_1, r_2, \cdots, r_n \in \mathbb{C})$

eg. $x^n - 1 = (x-1)(x-\zeta)(x-\zeta^2)(x-\zeta^3)\cdots(x-\zeta^{n-1})$ where $\zeta = e^{2\pi i/n}$.

de Moivre's formula: $\quad e^{i\theta} = \cos\theta + i\sin\theta$

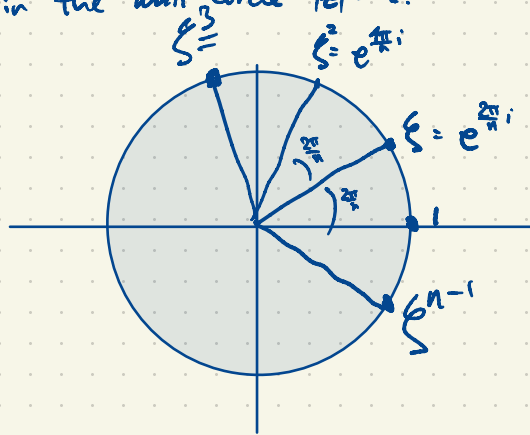Complex numbers $\mathbb{C} = \{a+bi : a,b \in \mathbb{R}\}$, $i = \sqrt{-1}$

Every $z \in \mathbb{C}$ has unique representation as $z = a+bi$ $(a,b \in \mathbb{R})$ in rectangular coordinates

$z = a+bi = re^{i\theta}$

$a = \text{Re } z = $ real part of z
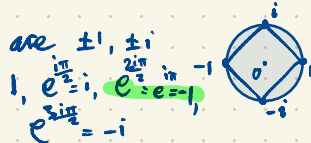$b = \text{Im } z = $ imaginary part of z.
$r = |z| = \sqrt{a^2+b^2}$

The roots of $x^n - 1$ are the nth roots of unity : $1, \zeta, \zeta^2, \cdots, \zeta^{n-1}$ forming the vertices of a regular n-gon inscribed in the unit circle $|z| = 1$.

ζ^3
$\zeta^2 = e^{\frac{4\pi}{n}i}$
$\zeta = e^{\frac{2\pi}{n}i}$
$\frac{2\pi}{n}$
1
ζ^{n-1}

Eg. $n=4$
The fourth roots of unity are $\pm 1, \pm i$

Euler's Formula $e^{i\pi} = -1$

$1, e^{\frac{i\pi}{2}} = i, \quad e^{i\pi} = e = -1,$
$e^{\frac{3i\pi}{2}} = -i$

$e^{i\pi} + 1 = 0$

Eg. $n=3$: The three cube roots of unity in $\mathbb{C}$ are $1, \omega, \omega^2$ where
$\omega = e^{\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
$x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$
$\omega = \frac{-1 \pm \sqrt{3}}{2}$

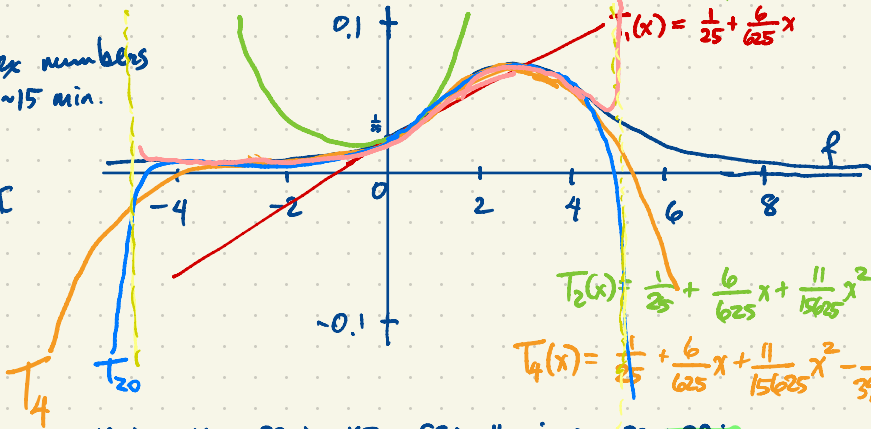$\omega^2 = \bar{\omega}$

follow links on course website
instructional videos $\longrightarrow$ complex numbers
~15 min.

Eg. consider $f(x) = \dfrac{1}{x^2 - 6x + 25}$

This function has poles at $x = 3 \pm 4i \in \mathbb{C}$

with $|3 \pm 4i| = 5$

$T_1(x) = \dfrac{1}{25} + \dfrac{6}{625}x$

$T_2(x) = \dfrac{1}{25} + \dfrac{6}{625}x + \dfrac{11}{15625}x^2$

$T_4(x) = \dfrac{1}{25} + \dfrac{6}{625}x + \dfrac{11}{15625}x^2 - \dfrac{84}{390625}x^3 - \dfrac{779}{9765625}x^4$

By the Binomial Theorem

$(1+i)^{11} = 1 + 11i - 55 - 165i + 330 + 462i - 462 - 330i + 165 + 55i - 11 - i = -32 + 32i$
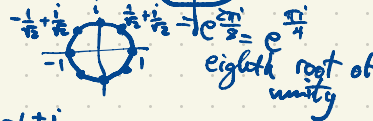
Much faster way to evaluate powers $z^n = (x+iy)^n = x^n + nx^{n-1}yi + \cdots + i^n y^n$ (Binomial Theorem)

$1 + i = \sqrt{2}\, e^{\frac{i\pi}{4}}$

$|1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$(1+i)^{11} = \left(\sqrt{2}\, e^{\frac{i\pi}{4}}\right)^{11} = 32\sqrt{2}\, e^{\frac{i\,11\pi}{4}}$

$= 32\sqrt{2} \cdot \left(\dfrac{-1+i}{\sqrt{2}}\right)$

$= -32 + 32i$

n^{th} roots of $z = re^{i\theta}$, $r = |z|$

all complex numbers whose n^{th} power is z

$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{i\theta}{n}},\ r^{\frac{1}{n}} e^{\frac{i\theta}{n} + \frac{2\pi i}{n}},\ r^{\frac{1}{n}} e^{\frac{i(\theta + 4\pi)}{n}}, \ldots, r^{\frac{1}{n}} e^{i\frac{\theta + 2(n-1)\pi}{n}}$

i.e. $r^{\frac{1}{n}} e^{i\frac{\theta + 2k\pi}{n}}$, $k = 0, 1, 2, \ldots, n-1$

$\zeta = \dfrac{1+i}{\sqrt{2}}$

$\zeta'' = \zeta^3 = \dfrac{-1+i}{\sqrt{2}}$

eighth root of unity

Cube roots of unity in $\mathbb{C}$: $1, \omega, \omega^2 = \bar{\omega}$

$$\omega = e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{3}}{2} = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$$

$$\omega^2 + \omega + 1 = 0$$
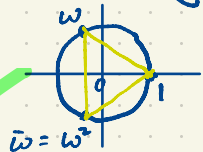
$\bar{\omega} = \omega^2$ $\qquad$ ω is a root of $x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$

$$\tau(\omega) = \omega^2$$
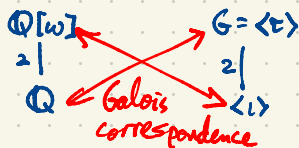
$G = \text{Aut } \mathbb{Q}[\omega] = \langle \tau \rangle = \{1, \tau\}$
where $\tau(z) = \bar{z}$

Now let $\alpha = \sqrt[3]{2}$, $\quad F = \mathbb{Q}[\alpha]$.
The min. poly. of α over $\mathbb{Q}$ is $x^3 - 2 \in \mathbb{Q}[x]$.

$$F = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}, \qquad \text{Aut } F = \{1\}$$

$3|$
$\mathbb{Q}$

$$x^3 - 2 = (x-\alpha)(x^2 + \alpha x + \alpha^2)$$

The other roots of $x^3 - 2$ are not in $F = \mathbb{Q}[\alpha]$ ie.
the extension $F \supset \mathbb{Q}$ is not normal.

Scale by factor of α

$$x^3 - 2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \text{ where } \alpha_1 = \alpha, \quad \alpha_2 = \alpha\omega, \quad \alpha_3 = \alpha\omega^2$$

$\alpha^2 = 2$

$(\alpha\omega)^3 = \alpha^3\omega^3 = 2 \cdot 1 = 2$

$(\alpha\omega^2)^3 = \alpha^3\omega^6 = 2 \cdot 1 = 2$

$x^3 - 2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$
$\qquad = (x - \alpha)(x - \alpha\omega)(x - \alpha\omega^2)$

basis $\{1, \omega\}$ so $[E:F] = 2$

basis $1, \alpha, \alpha^2$ so $[F:\mathbb{Q}] = 3$

$$E \supset F \supset \mathbb{Q}$$
$\mathbb{Q}[\alpha_1, \alpha_2, \alpha_3] \quad \mathbb{Q}[\alpha]$
$\mathbb{Q}[\alpha, \omega]$

$[E : \mathbb{Q}] = 2 \cdot 3 = 6$

$\omega = \frac{1}{2}\alpha_1^2 \alpha_2$
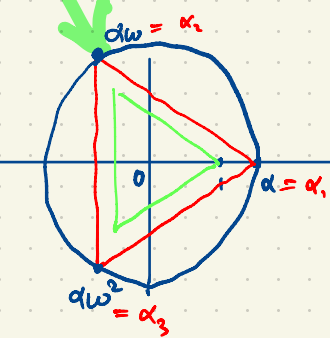$= \frac{1}{2} \cdot 2^{2/3} \cdot 2^{1/3}\omega$
$= \omega$

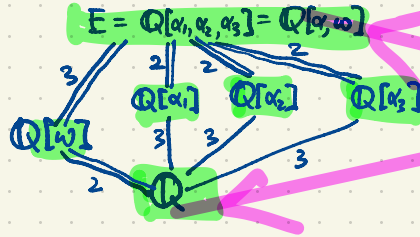There are $3! = 6$ permutations of $\alpha_1, \alpha_2, \alpha_3$.

x	$\sigma(x)$	$\tau(x)$
α_1	α_2	α_1
α_2	α_3	α_3
α_3	α_1	α_2
α	$\alpha\omega$	α
ω	ω	ω^2

$\tau(x) \leftarrow$ complex conjugation

In $S_3 = \langle \sigma, \tau \rangle$, $\sigma = (123)$, $\tau = (23)$.

$\sigma(\omega) = \sigma\left(\frac{\alpha_2}{\alpha_1}\right) = \frac{\sigma(\alpha_2)}{\sigma(\alpha_1)} = \frac{\alpha_3}{\alpha_2} = \frac{\alpha\omega^2}{\alpha\omega} = \omega$
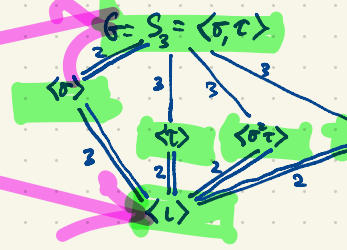
$\sigma\left(\frac{\alpha\omega}{\alpha}\right)$

$\tau(\omega) = \tau\left(\frac{\alpha_2}{\alpha_1}\right) = \frac{\alpha_3}{\alpha_1} = \frac{\alpha\omega^2}{\alpha} = \omega^2 = \bar{\omega}$

$E = \mathbb{Q}[\alpha_1, \alpha_2, \alpha_3] = \mathbb{Q}[\alpha, \omega]$

Hasse diagram of subfields of E

$G = S_3 = \langle \sigma, \tau \rangle$

Hasse diagram of subgroups of $G = \text{Aut } E$

$\sigma^2 \tau = \tau \sigma$

$\sigma \tau = \tau \sigma^2$

Double lines indicate normality.

$\mathbb{Q}[\alpha_1] \quad \mathbb{Q}[\alpha_2] \quad \mathbb{Q}[\alpha_3]$

$\mathbb{Q}[\omega]$

$\langle \sigma \rangle$

$\langle \tau \rangle \quad \langle \sigma^2 \tau \rangle \quad \langle \sigma \tau \rangle$

Using right-to-left composition

Galois correspondence

$\sigma^2 \tau = (132)(23) = (13)$

$\sigma \tau = (123)(23) = (12)$

$\tau = (23)$

A subgroup $H \leq G$ is __normal__ if its left and right cosets agree i.e. $gH = Hg$ for all $g \in G$.

Eg. in $G = S_3$, $H = \langle \sigma \rangle = \langle (123) \rangle$ is normal.

eg. $(12)H = (12)\{(), (123), (132)\} = \{(12), (23), (13)\}$

$\quad\quad\quad\quad\quad\quad\quad \underset{id}{\uparrow} \quad\quad \underset{\sigma}{\uparrow} \quad\quad \underset{\sigma^2}{\uparrow}$

$(12)(123) = (1)(23)$
$\quad\quad\quad\quad = (23)$

$H(12) = \{(), (123), (132)\}(12) = \{(12), (13), (23)\}$

$\langle \tau \rangle$ is a subgroup of G which is __not__ normal in G.

$(13)\langle \tau \rangle = (13)\{(), (23)\} = \{(13), (132)\}$

$\langle \tau \rangle (13) = \{(), (23)\}(13) = \{(13), (123)\}$

The extension $\mathbb{Q}[\alpha] \supset \mathbb{Q}$ of degree $[\mathbb{Q}[\alpha] : \mathbb{Q}] = 3$ is __not__ normal
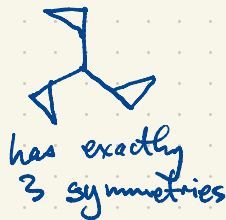
$\overset{?}{\mathbb{Q}}[\alpha_1]$

since the min. poly. of α over $\mathbb{Q}$ is $x^3 - 2$ with $\mathbb{Q}[\alpha]$ containing only one of the three roots of $x^3 - 2$.

Galois theory handout : ignore "separable" for now.

Example of an extension $E \supset \mathbb{Q}$ of degree 3 with $G = \text{Aut } E$ of order 3?

$f(x) = x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x]$ is irreducible

$f(x) = (x - \alpha)(x - \beta)(x - \gamma)$ where $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$

$\alpha^3 = 1 + 2\alpha - \alpha^2$

$\alpha^4 = \alpha + 2\alpha^2 - \alpha^3 = \alpha + 2\alpha^2 - (1 + 2\alpha - \alpha^2) = -1 - \alpha + 3\alpha^2$

$\alpha^5 = 3 + 5\alpha - 4\alpha^2$

$\alpha^6 = -4 - 5\alpha + 9\alpha^2$

has exactly 3 symmetries

Check that $\alpha^2 - 2$ is also a root of $f(x)$:

$f(\alpha^2 - 2) = (\alpha^2 - 2)^3 + (\alpha^2 - 2)^2 - 2(\alpha^2 - 2) - 1 = 0$ after collecting terms, so $\alpha^2 - 2 \in \{\alpha, \beta, \gamma\}$.

Can $\alpha^2 - 2 = \alpha$? No. If α is a root of $f(x) = x^3 + x^2 - 2x - 1$ and a root of $g(x) = x^2 - x - 2$ then α is a root of

$$\gcd(f(x), g(x)) = r(x) f(x) + s(x) g(x) \text{ by Euclid's Algorithm}$$

which is a factor of $f(x)$ of degree less than 3, a contradiction.

WLOG $\beta = \alpha^2 - 2$. Now $\beta^2 - 2$ is also a root of $f(x)$ by the same reasoning, so $\beta^2 - 2 \in \{\alpha, \beta, \gamma\}$.

As before, $\beta^2 - 2 \neq \beta$. If $\beta^2 - 2 = \alpha$ then $(\alpha^2 - 2)^2 - 2 = \alpha = \alpha^4 - 4\alpha^2 + 4 - 2 = \alpha$

$$\alpha^4 - 4\alpha^2 - \alpha + 2 = 0$$

but $\gcd(x^3 + x^2 - 2x - 1, x^4 - 4x^2 - x + 2) = 1$, contradiction.
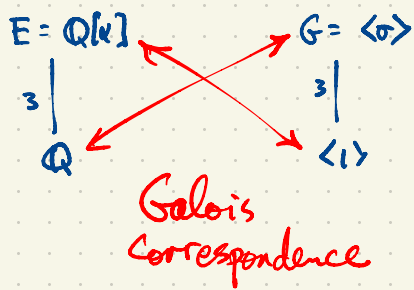
Better: $-1 - \alpha + 3\alpha^2 - 4\alpha^2 - \alpha + 2 = 0$

$1 - 2\alpha - \alpha^2 = 0$

So $\beta^2 - 2 = \gamma$. Now $\gamma^2 - 2 = \alpha$. Indeed

$\gamma^2 - 2 = (\beta^2 - 2)^2 - 2 = ((\alpha^2 - 2)^2 - 2)^2 - 2 = \alpha$.

The map $x \mapsto x^2 - 2$ gives a cyclic permutation $\sigma: \alpha \mapsto \beta \mapsto \gamma \mapsto \alpha$. cyclic

The field $E = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$ of degree $[E : \mathbb{Q}] = 3$ (a cubic extension)

has automorphism group $G = \text{Aut } E = \langle \sigma \rangle = \{\iota, \sigma, \sigma^2\}$, cyclic of order 3.

$$E = \mathbb{Q}[\kappa] \qquad G = \langle \sigma \rangle$$

$$3 \Big| \qquad\qquad 3 \Big|$$

$$\mathbb{Q} \qquad\qquad \langle 1 \rangle$$

Galois
Correspondence

Exercise : Find three 3×3 matrices over $\mathbb{Q}$
  A, B, C which are roots of $f(x)$
  Satisfying $B = A^2 - 2I$
(not HW) $\qquad C = B^2 - 2I$
  $\qquad\qquad A = C^2 - 2I$

$$A^3 + A^2 - 2A - I = 0$$
$$B^3 + B^2 - 2B - I = 0$$
$$C^3 + C^2 - 2C - I = 0.$$