Math 4520

Fields

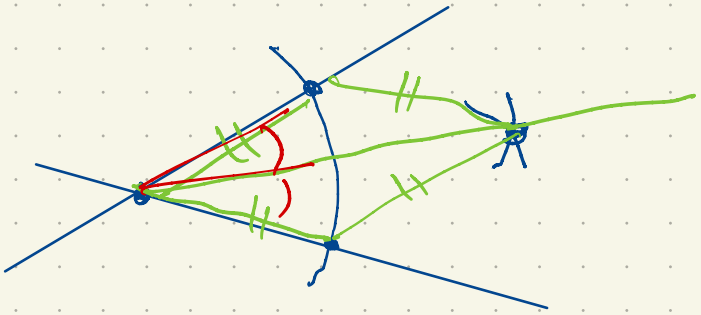
Book 3

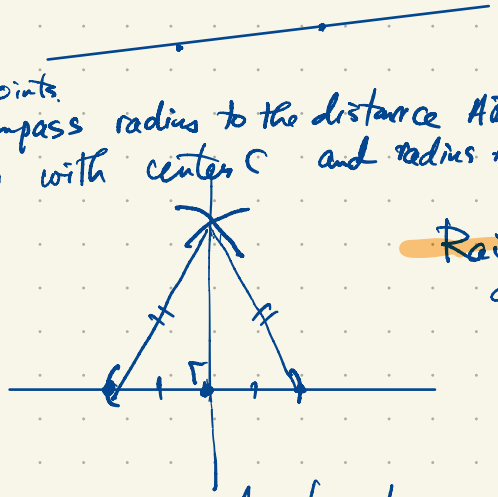
Suppose $\theta \in \mathbb{C}$ is a root of some nonzero poly $m(x) \in \mathbb{Q}[x]$.
Then take the smallest degree such $m(x)$. Assuming $m(x)$ is monic, this is the minimal polynomial of θ (over $\mathbb{Q}$). It is irreducible in $\mathbb{Q}[x]$.
Conversely, if $m(x) \in \mathbb{Q}[x]$ is a monic irreducible poly. having θ as a root, then it is the minimal poly. of θ. (over $\mathbb{Q}$).
To find $m(x)$: look at $1, \theta, \theta^2, \theta^3, \dots$ until you find a power θ^n which is a linear combination of smaller powers $1, \theta, \theta^2, \dots, \theta^{n-1}$.
$\theta^n = a_0 \cdot 1 + a_1\theta + a_2\theta^2 + \dots + a_{n-1}\theta^{n-1}$ $\iff$ θ is a root of $x^n - a_{n-1}x^{n-1} - \dots - a_1 x - a_0 \in \mathbb{Q}[x]$ (monic).

Straightedge and Compass Constructions

Here we allow certain operations:
Given two marked points on a page, we can mark the unique line through those two points.
Given any pair of points A, B we can set the compass radius to the distance AB and given a further point C, we can then mark the circle (or any arc thereof) with center C and radius AB.

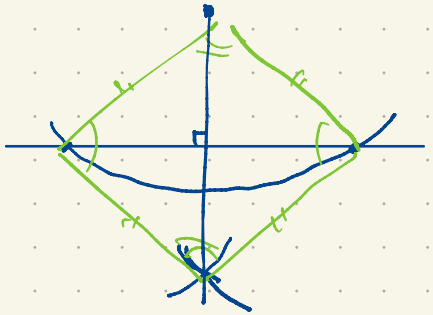
Using these tools, we can bisect any angle:



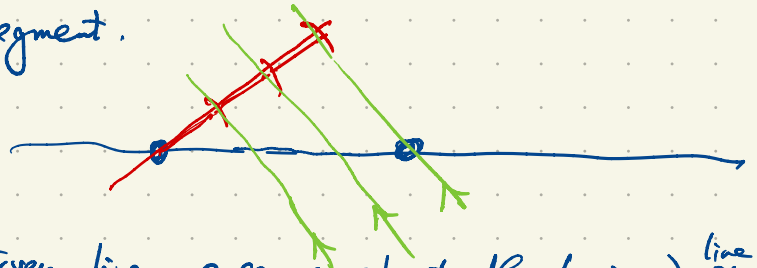


Raise a perpendicular to any line through any given point on that line

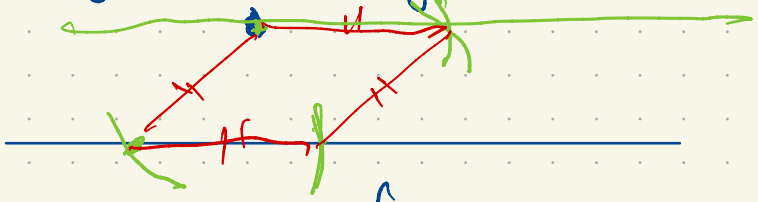
Drop a perpendicular to a line from any point not on the line:

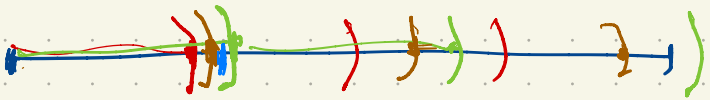


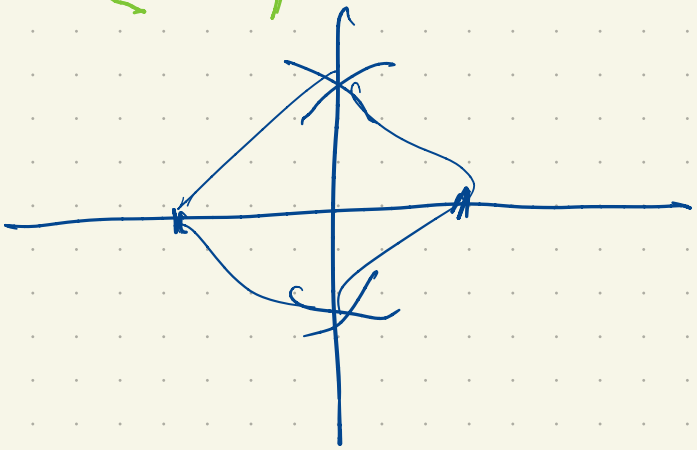
We can trisect any line segment.

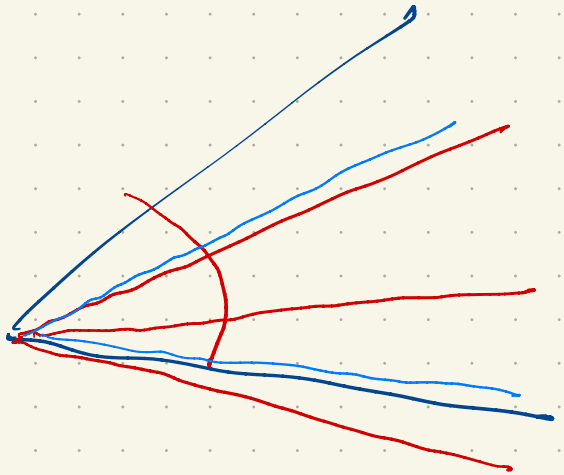


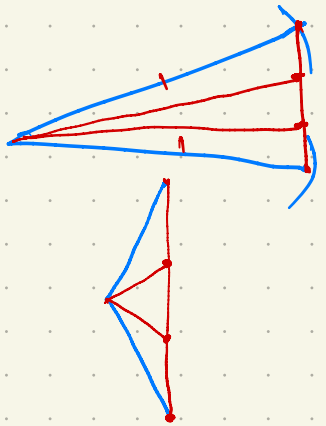
Through any point not on a given line, we can construct the (unique) line parallel to the given line.



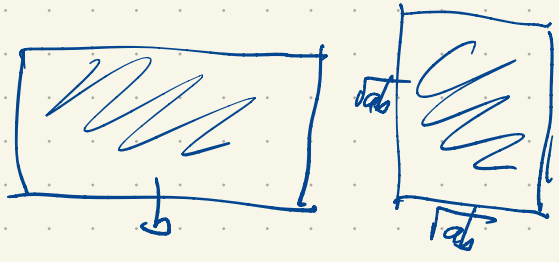




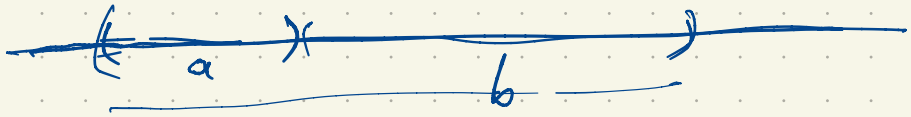


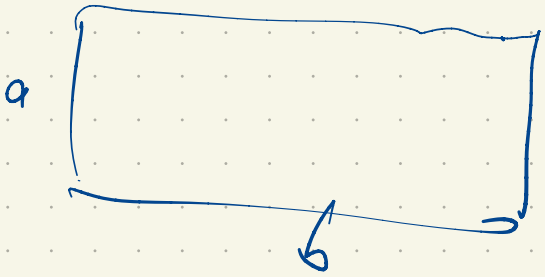


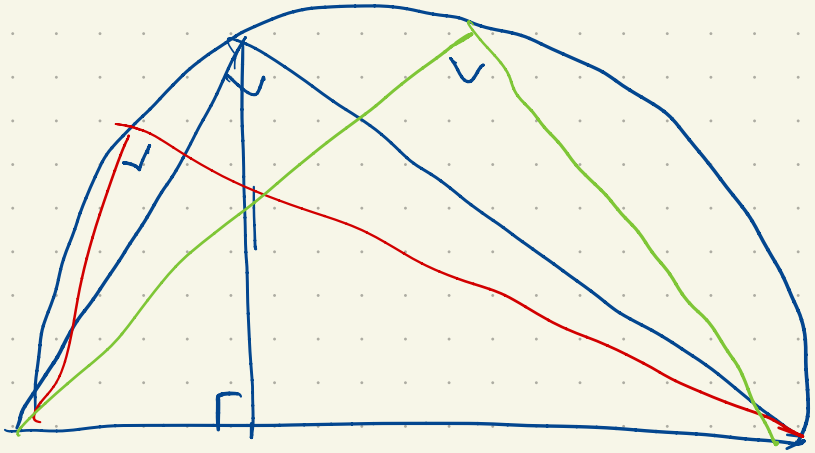
Given line segments of length a and b, we can construct a line segment of length $\sqrt{ab}$ (the geometric mean of a and b).

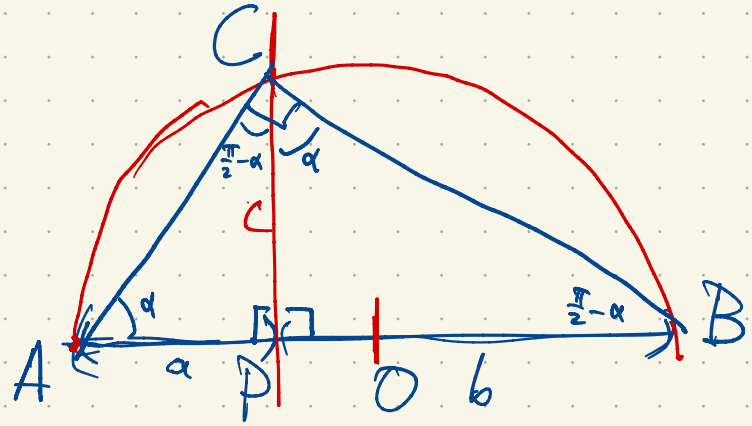


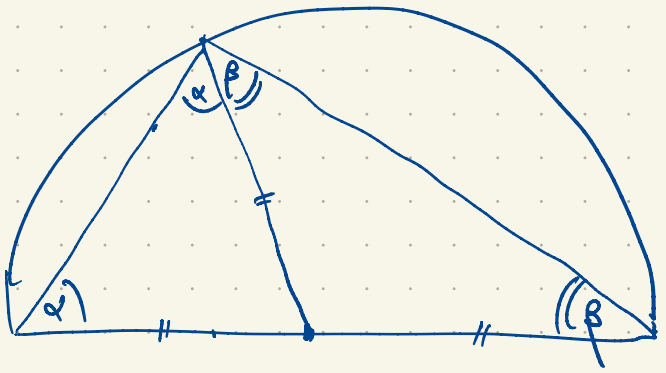
Given a and b,











$\alpha + (\alpha+\beta) + \beta = \pi$

$\alpha+\beta = \frac{\pi}{2}$

Triangle similar

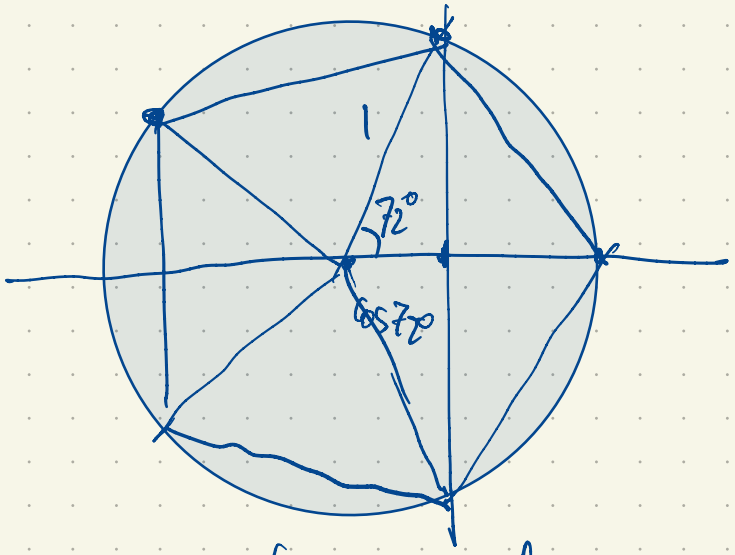
$\triangle APC \sim \triangle CPB$

by comparing corresponding angles $\alpha, \frac{\pi}{2}, \frac{\pi}{2}-\alpha$

$\frac{AP}{PC} = \frac{PC}{PB}$ i.e. $\frac{a}{c} = \frac{c}{b} \Rightarrow c^2 = ab$

$\Rightarrow c = \sqrt{ab}$.

Using our basic constructions (above), we can construct a regular pentagon.



circle of radius 1.

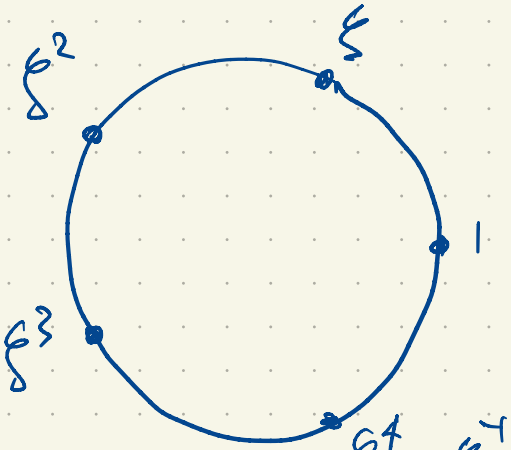


Construct a line segment of length $\cos 72° = \cos\frac{2\pi}{5}$.



$\zeta = e^{2\pi i/5}$, $\zeta^5 = 1$.

$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$



$\zeta^4 = \zeta^{-1} = \bar{\zeta} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

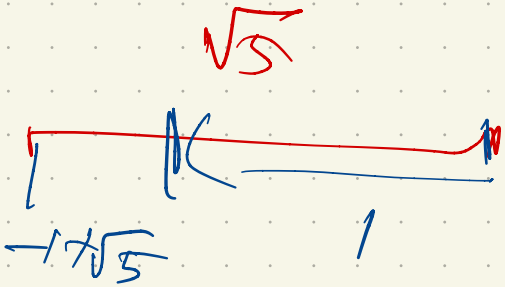
$2\cos\frac{2\pi}{5} = \zeta + \zeta^{-1} = \alpha$ in $\mathbb{Q}[\zeta] \supset \mathbb{Q}$ extension of degree 4

ζ is a root of $x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$

so ζ is a root of $x^4+x^3+x^2+x+1$ which is irreducible; it is the min. poly. of ζ, i.e. $\zeta^4+\zeta^3+\zeta^2+\zeta+1 = 0$.

$\alpha = \zeta + \zeta^{-1}$

$\alpha^2 = \zeta^2 + \zeta^{-2} + 2$



$\alpha + \alpha^2 = 2 + \zeta + \zeta^2 + \zeta^4 + \zeta^3$

$= 1 + (1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4)$

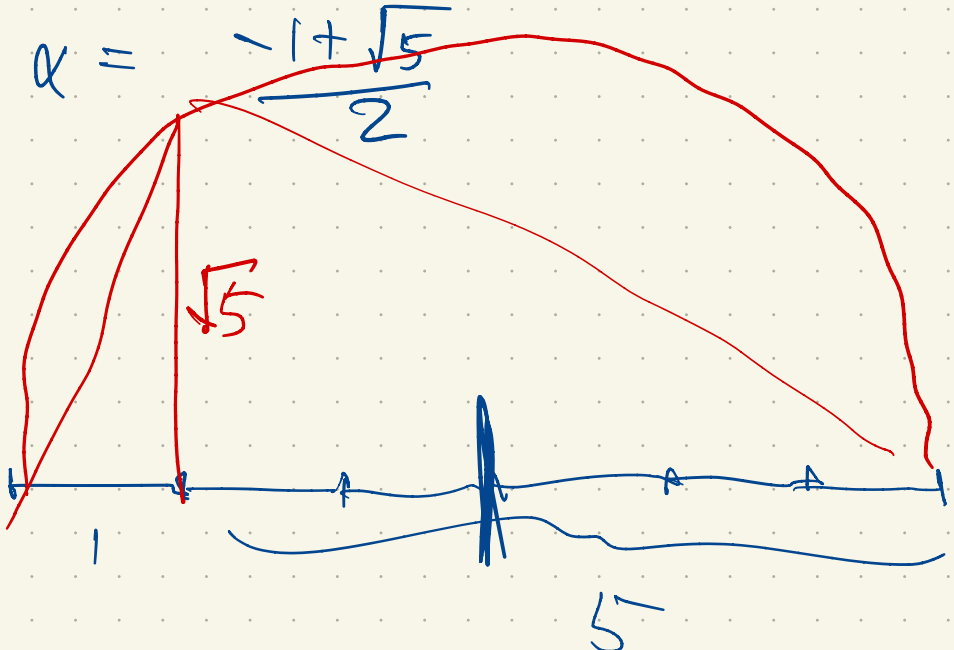
0

$\alpha^2 + \alpha - 1 = 0$

$\alpha = \dfrac{-1 \pm \sqrt{5}}{2}$

but $\alpha = 2\cos\dfrac{2\pi}{5} > 0$

so $\alpha = \dfrac{-1+\sqrt{5}}{2}$



For which n can we construct a regular n-gon? (using only straightedge and compass).

We know how to construct a regular n-gon for $n = 2^k \cdot 3^l \cdot 5^m \cdot 17^n \cdot 257^s \cdot 65537^t$ $k \geq 0$, $l, m, n, s, t \in \{0, 1\}$

$$3 = 2+1$$
$$5 = 2^2+1$$
$$17 = 2^4+1$$
$$257 = 2^8+1$$
$$65537 = 2^{16}+1$$

A regular n-gon is constructible using straightedge and compass iff n is a power of 2 times a product of distinct Fermat primes. A Fermat prime is a prime that is 1 more than a power of 2.

The only Fermat primes that are known are 3, 5, 17, 257, 65537.

We'll show that a regular 9-gon cannot be constructed using straightedge and compass o a 120° cannot be trisected.