Fields

Book I

Fields

Let F be a set containing distinct elements called 0 and 1 (thus $0 \neq 1$). Suppose addition, subtraction, multiplication and division are defined for all elements of F (except division by 0 is not defined).
Thus $a+b,\ a-b,\ ab,\ \frac{a}{d} \in F$ whenever $a, b, d \in F$ and $d \neq 0$.
Define $-a = 0 - a$.

If the following properties are satisfied by *all* elements $a, b, c, d \in F$ with $d \neq 0$, then F is a field.

$$a + b = b + a \qquad a + (b + c) = (a + b) + c \qquad ab = ba$$

$$a + 0 = a \qquad a(bc) = (ab)c \qquad 1a = a$$

$$a + (-a) = 0 \qquad a(b + c) = ab + ac \qquad \frac{a}{d}d = a$$

$$a + (-b) = a - b$$

$\mathbb{Q}^{2\times 2} = \{2\times 2 \text{ matrices over } \mathbb{Q}\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Q} \right\}$ is not a field.

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ identity

$A + 0 = A$, $AI = A = IA$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has no inverse. $A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = I$ has no solution for A.

Moreover, $AB \neq BA$ in general.

$\mathbb{Q}^{2\times 2}$ is a (non-commutative) ring with identity.

It has a subring $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a,d \in \mathbb{Q} \right\}$ is a commutative subring with identity.

But D is not a field since it has non-invertible elements.

D has zero divisors: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. A field can never have zero divisors.

(If d is a zero divisor then $cd = 0$ where $c,d \neq 0$ so $\left(\frac{c}{d}\right)d = c \neq 0$, contradiction)

$$0 \cdot \frac{1}{d} = \frac{0}{d} = \frac{cd}{d} =$$

For a commutative ring R with identity, being able to divide is stronger than having no zero divisors.

An example of a commutative ring with identity having no zero divisors but not a field (division fails in general) is $\mathbb{Z}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Eg. $F = \left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} : a,b \in \mathbb{Q} \right\} \subset \mathbb{Q}^{2\times 2}$ is a subring, containing $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If $\begin{bmatrix} a & b \\ 2b & a \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then $\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}^{-1} = \frac{1}{a^2-2b^2}\begin{bmatrix} a & -b \\ -2b & a \end{bmatrix}$ (Note: $a^2-2b^2 \neq 0$ since $\sqrt{2} \notin \mathbb{Q}$)

Why is F a commutative subring? Elements of F have the form

$\begin{bmatrix} a & b \\ 2b & a \end{bmatrix} = aI + bS$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ so $F = \{aI + bS : a,b \in \mathbb{Q}\}$ is the span of $\{I, S\}$ in $\mathbb{Q}^{2\times 2}$ (F is a 2-dimensional subspace of $\mathbb{Q}^{2\times 2}$, a 4-dimensional vector space).

$$(aI+bS)(cI+dS) = acI + (ad+bc)S + bdS^2 = (cI+dS)(aI+bS) \quad , \quad S^2 = \begin{bmatrix}0&1\\2&0\end{bmatrix}\begin{bmatrix}0&1\\2&0\end{bmatrix} = 2I$$

$$= (ac+2bd)I + (ad+bc)S$$

Compare: $K = \mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\}$. is a field.

$$(a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$$

$$(a+b\sqrt{2})(c+d\sqrt{2}) = ac + (ad+bc)\sqrt{2} + 2bd = (ac+2bd) + (ad+bc)\sqrt{2}$$

Note: $F \cong K$ (they are isomorphic)

An explicit isomorphism $\phi : K \to F$ is given by $\phi(a+b\sqrt{2}) = \begin{bmatrix}a&b\\2b&a\end{bmatrix} = aI+bS$.

ϕ is bijective

$$\phi(x+y) = \phi(x)+\phi(y)$$

$$\phi(xy) = \phi(x)\phi(y)$$

Similarly $\left\{\begin{bmatrix}a&b\\-b&a\end{bmatrix} : a,b \in \mathbb{R}\right\} \subset \mathbb{R}^{2\times 2}$ is a subring isomorphic to $\mathbb{C}$.

An isomorphism $\mathbb{C} \to \left\{\begin{bmatrix}a&b\\-b&a\end{bmatrix} : a,b\in\mathbb{R}\right\}$ is $a+bi \mapsto \begin{bmatrix}a&b\\-b&a\end{bmatrix}$ $(a,b \in \mathbb{R})$.

$$\mathbb{Q}[\sqrt{2}] = \{ a+b\sqrt{2} : a,b \in \mathbb{Q} \}$$

$$\alpha = 5+3\sqrt{2}, \quad \beta = 7-\sqrt{2}$$

$$\alpha+\beta = 12+2\sqrt{2}$$

$$\alpha-\beta = -2+4\sqrt{2}$$

$$\alpha\beta = (5+3\sqrt{2})(7-\sqrt{2}) = 35-5\sqrt{2}+21\sqrt{2}-6 = 29+16\sqrt{2}$$

$$\frac{\alpha}{\beta} = \frac{5+3\sqrt{2}}{7-\sqrt{2}} = \frac{5+3\sqrt{2}}{7-\sqrt{2}} \cdot \frac{7+\sqrt{2}}{7+\sqrt{2}} = \frac{35+5\sqrt{2}+21\sqrt{2}+6}{47} = \frac{41+26\sqrt{2}}{47} = \frac{41}{47} + \frac{26}{47}\sqrt{2}$$

Alternatively, $\frac{\alpha}{\beta} = \alpha\beta^{-1}$

in matrix representation: $\underbrace{\begin{bmatrix} 5 & 3 \\ 6 & 5 \end{bmatrix}}_{\alpha} \cdot \frac{1}{47}\begin{bmatrix} 7 & 1 \\ 2 & 7 \end{bmatrix} = \frac{1}{47}\begin{bmatrix} 41 & 26 \\ 52 & 41 \end{bmatrix}$

$$\beta \longmapsto \begin{bmatrix} 7 & -1 \\ -2 & 7 \end{bmatrix}$$

$$\beta^{-1} \longmapsto \frac{1}{47}\begin{bmatrix} 7 & 1 \\ 2 & 7 \end{bmatrix}$$

Similar: $\mathbb{Q}[\sqrt[3]{2}] = \mathbb{Q}[\theta]$, $\theta = \sqrt[3]{2}$.

$\{ a+b\theta : a,b \in \mathbb{Q} \}$ is not a field, not even a ring since it's not closed under multiplication.

$\mathbb{Q}[\theta] = \{ a+b\theta+c\theta^2 : a,b,c \in \mathbb{Q} \}$. <u>is</u> a field.

$$\theta^3 = 2$$
$$\theta^4 = 2\theta$$
$$\theta^5 = 2\theta^2$$
$$\theta^6 = 4$$

$$\alpha = 5+3\theta$$
$$\beta = 7-\theta$$

$$\alpha+\beta = 12+2\theta$$
$$\alpha-\beta = -2+4\theta$$
$$\alpha\beta = (5+3\theta)(7-\theta) = 35-5\theta+21\theta-3\theta^2$$
$$= 35+16\theta-3\theta^2$$

$$\frac{\alpha}{\beta} = \frac{5+3\theta}{7-\theta} = \boxed{a} + \boxed{b}\theta + \boxed{c}\theta^2 = \frac{251}{341} + \frac{182}{341}\theta + \frac{26}{341}\theta^2 = \frac{1}{341}(251 + 182\theta + 26\theta^2)$$

rational coefficients

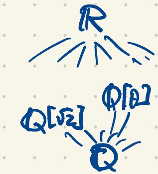
$a, b, c \in \mathbb{Q}$

$\theta^3 = 2$

$\theta^3 - 2 =$

$\theta = \sqrt[3]{2}$

θ is a root of $x^3 - 2 = (x-\theta)(x^2 + \theta x + \theta^2)$



$$5 + 3\theta = (a + b\theta + c\theta^2)(7-\theta)$$
$$= 7a + (7b-a)\theta + (7c-b)\theta^2 - 2c$$
$$= (7a-2c) + (7b-a)\theta + (7c-b)\theta^2$$

Hopefully

$$\begin{aligned} 7a \quad\quad -2c &= 5 \\ -a + 7b \quad\quad &= 3 \\ -b + 7c &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 7 & 0 & -2 & 5 \\ -1 & 7 & 0 & 3 \\ 0 & -1 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 49 & -2 & 26 \\ -1 & 7 & 0 & 3 \\ 0 & -1 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -7 & 0 & -3 \\ 0 & 49 & -2 & 26 \\ 0 & 1 & -7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -7 & 0 & -3 \\ 0 & 1 & -7 & 0 \\ 0 & 49 & -2 & 26 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -49 & -3 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 341 & 26 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -49 & -3 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 1 & \frac{26}{341} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{251}{341} \\ 0 & 1 & 0 & \frac{182}{341} \\ 0 & 0 & 1 & \frac{26}{341} \end{array}\right]$$

$$\begin{array}{r} 26 \\ \times\ 7 \\ \hline 182 \end{array} \qquad \begin{array}{r} 49 \\ \times\ 26 \\ \hline 294 \\ 98\ \\ \hline 1274 \end{array} \qquad \begin{array}{r} 341 \\ \times\ 3 \\ \hline 1023 \end{array}$$

$$-3 + 49 \cdot \frac{26}{341}$$
$$= -3 + \frac{1274}{341}$$
$$= \frac{-1023 + 1274}{341} = \frac{251}{341}$$

Check: $\frac{1}{341}(251 + 182\theta + 26\theta^2)(7-\theta) = \frac{1}{341}(1757 + 1023\theta + 0\theta^2 - 52)$
$$= \frac{1}{341}(1705 + 1023\theta)$$
$$= 5 + 3\theta \quad \checkmark$$

$\mathbb{Q}[\theta]$ is a cubic field extension of $\mathbb{Q}$: it is a 3-dimensional vector space over $\mathbb{Q}$, with basis $1, \theta, \theta^2$.

Alternatively, use 3×3 matrices to represent elements of $\mathbb{Q}[\theta]$.

Take $T = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ to represent θ. $T^3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I$

$$E = \{ aI + bT + cT^2 : a,b,c \in \mathbb{Q} \} = \underbrace{\left\{ \begin{bmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{bmatrix} : a,b,c \in \mathbb{Q} \right\}}_{\text{This subring is a field.}} \subset \underbrace{\mathbb{Q}^{3\times 3}}_{\text{noncommutative ring with identity having zero divisors}}$$

$\mathbb{Q}[\theta] \cong E$ via the isomorphism

$$\begin{array}{ccc} \mathbb{Q}[\theta] & & E \\ \Psi & & \Psi \\ a+b\theta+c\theta^2 & \longmapsto & aI+bT+cT^2 \end{array}$$

Are there any fields "between" $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{2}]$, or between $\mathbb{Q}$ and $\mathbb{Q}[\theta]$?

Are there any fields "between" $\mathbb{R}$ and $\mathbb{C}$?

Suppose $\mathbb{R} \subset F \subset \mathbb{C}$ is a tower of fields (F is a subfield of $\mathbb{C}$ and $\mathbb{R}$ is a subfield of F).

'$\subset$' always means strict containment in this course.

$\subseteq$ vs $\subset$

$\leq$ vs $<$

Since $F \supset \mathbb{R}$, there exists $\alpha \in F$, $\alpha \notin \mathbb{R}$. Then $\alpha, 1$ are linearly independent over $\mathbb{R}$, i.e. $\alpha \neq a\cdot 1$ for any $a \in \mathbb{R}$. However $\mathbb{C}$ is 2-dimensional over $\mathbb{R}$ with basis $1, i$ (every complex number is uniquely expressible as $z = a\cdot 1 + b\cdot i$ with $a,b \in \mathbb{R}$). So $1, \alpha$ is a basis for F. So $F = \mathbb{C}$.

Is there any field extension $\mathbb{C} \subset F$ with F 2-dimensional over $\mathbb{C}$?
No, but there do exist fields $F \supset \mathbb{C}$ which are infinite-dimensional extensions.

Consider the ring $\mathbb{C}[x] = \{$polynomials in x with complex coefficients$\}$
$= \{ a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n : a_i \in \mathbb{C},\ n \geq 0 \}$

This is a ring but not quite a field eg.

$$\frac{5 + 7x + ix^2}{3 - (4+i)x + 43x^2} \notin \mathbb{C}[x]$$

$\mathbb{C}(x) =$ field of fractions of $\mathbb{C}[x]$
$=$ field of rational functions in x with complex coefficients

Just like constructing $\mathbb{Q}$ from $\mathbb{Z}$.
Another example of this: We'll construct a countably infinite subfield of $\mathbb{R}$ containing π.
This contains the subring $\mathbb{Q}[\pi] = \{ a_0 + a_1\pi + a_2\pi^2 + \dots + a_n\pi^n : n \geq 0,\ a_i \in \mathbb{Q} \}$
$\pi \in \mathbb{Q}[\pi]$ has no (multiplicative) inverse in $\mathbb{Q}[\pi]$ since if

$$1 = \pi (a_0 + a_1\pi + a_2\pi^2 + \dots + a_n\pi^n) \quad a_i \in \mathbb{Q},\ n \geq 0,$$

a contradiction since π is transcendental. (π would be a root of a nonzero polynomial $a_n x^{n+1} + a_{n-1} x^n + \dots + a_2 x^3 + a_1 x^2 + a_0 x - 1$)
(Lindemann 1800's)

$\mathbb{Q}(\pi) = \{ \frac{a}{b} : a, b \in \mathbb{Q}[\pi],\ b \neq 0 \}$ is the field of quotients of the ring $\mathbb{Q}[\pi]$

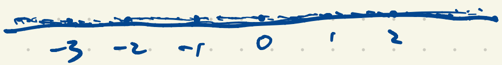
$\mathbb{Q}(\sqrt{2}) = \{ \frac{a}{b} : a, b \in \mathbb{Q}[\sqrt{2}],\ b \neq 0 \} = \mathbb{Q}[\sqrt{2}]$ is already a field. $\sqrt{2}$ is algebraic: it is a root of a nonzero poly. $x^2 - 2 \in \mathbb{Q}[x]$
Every $\alpha \in \mathbb{C}$ is either algebraic or transcendental, never both.

$\underset{\text{countable}}{\mathbb{Q}} \subset \underset{\text{uncountable}}{\mathbb{R}} \subset \underset{\text{uncountable}}{\mathbb{C}}$

$\underbrace{\mathbb{Q} \subset \mathbb{A}}_{\text{countable}} \subset \underset{\text{uncountable.}}{\mathbb{C}}$

$\mathbb{A} = \{\text{algebraic numbers}\}.$

$$\begin{array}{ccccc} & & \mathbb{A} & & \\ & \subset & & \subset & \\ \mathbb{Q} \subset \mathbb{A}\cap\mathbb{R} & & & & \mathbb{C} \\ & \subset & & \subset & \\ & & \mathbb{R} & & \end{array}$$



$\mathbb{Q} = \{a_1, a_2, a_3, a_4, \dots\}$

$$\begin{array}{cccc} a_1 + a_1 x & a_1 + a_2 x & a_1 + a_3 x & \cdots \\ a_2 + a_1 x & a_2 + a_2 x & a_2 + a_3 x & \cdots \\ a_3 + a_1 x & a_3 + a_2 x & a_3 + a_3 x & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

$\mathbb{Q}[\pi]$ is a countably infinite ring

so $\mathbb{Q}(\pi)$ is a countably infinite field.

Elements of $\mathbb{Q}(\pi) \subset \mathbb{R}$ look like

$$\frac{53.8\pi^3 - 17\pi + \frac{53}{7}}{42\pi^2 + 119\pi + \frac{103}{648}}$$

Compare: $\mathbb{Q}(e) \subset \mathbb{R}$, another countable subfield of $\mathbb{R}$.

Actually $\mathbb{Q}(e) \cong \mathbb{Q}(\pi)$. An ~~isomorphism~~ isomorphism is $f(e) \longmapsto f(\pi)$ where $f(x) \in \mathbb{Q}(x)$.

$\cong \mathbb{Q}(x)$ (x being an indeterminate ie. an {abstract / general / generic} symbol)

$\mathbb{Q}(x) \longrightarrow \mathbb{Q}(\pi)$ evaluation

$\mathbb{Q}(x) \longrightarrow \mathbb{Q}(e)$

$\mathbb{Q}(x) \longrightarrow \mathbb{Q}(\sqrt{2})$ doesn't quite work eg. the image of $\frac{x^3 + 7x^2 - 3}{x^2 - 2} \in \mathbb{Q}(x)$ is undefined; you can't evaluate this at $\sqrt{2}$.

But the evaluation maps at $\pi, e, \sqrt{2}, \dots$

$\left.\begin{array}{l} \mathbb{Q}[x] \longrightarrow \mathbb{Q}[\pi] \\ \mathbb{Q}[x] \longrightarrow \mathbb{Q}[e] \\ \mathbb{Q}[x] \longrightarrow \mathbb{Q}[\sqrt{2}] \\ \quad\vdots \end{array}\right\}$ all well-defined ring homomorphisms.

If $\phi: R \to S$ where R, S are rings, we say ϕ is a ring homomorphism if
$$\left.\begin{array}{l}\phi(a+b) = \phi(a) + \phi(b) \\ \phi(ab) = \phi(a)\phi(b)\end{array}\right\} \text{ for all } a,b \in R$$
We don't necessarily require $\phi(1) = 1$; and in general the rings R, S may not have identity.
If R, S are rings with identity ($1_R \in R$, $1_S \in S$) we might consider only homomorphisms of rings with identity i.e. $\phi(1_R) = \phi(1_S)$.

* Suppose F, K are fields. If $\phi: F \to K$ is a ring homomorphism then either
(i) $\phi(F) = \{0\}$ i.e. $\phi(a) = 0$ for all $a \in F$, (trivial) or
(ii) ϕ is one-to-one i.e. $\phi(F) \subseteq K$ is a subfield isomorphic to F.

Any homomorphism $\mathbb{Q}(x) \to \mathbb{R}$ is either trivial or it has the form $\mathbb{Q}(x) \to \mathbb{Q}(a)$, $f(x) \mapsto f(a)$ is an evaluation at some transcendental number $a \in \mathbb{R}$.

We have ring homomorphisms $\mathbb{Q}[x] \to \mathbb{C}^{n\times n}$ ($n \times n$ complex matrices) where we evaluate at a matrix $A \in \mathbb{C}^{n\times n}$, i.e. $f(x) \mapsto f(A)$
$$\tfrac{47}{3}x^2 + \tfrac{18}{11}x - \tfrac{41}{7} \mapsto \tfrac{47}{3}A^2 + \tfrac{18}{11}A - \tfrac{41}{7}I$$

(*) In a field F, every ideal is either $\{0\}$ or F.

An automorphism of a field F is an isomorphism $\phi: F \to F$. Eg bijective with
(i) Automorphisms of $\mathbb{Q}[\sqrt{2}]$? We want $\phi: \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{2}]$
$$\phi(a+b) = \phi(a) + \phi(b), \quad \phi(ab) = \phi(a)\phi(b).$$
- The identity $\phi(a) = a$
- $\phi(x) = x + t$, $t \in \mathbb{Q}[\sqrt{2}]$. These are not automorphism.

If $\phi : F \to F$ is any automorphism of a field F then

$\phi(0) = \phi(0+0) = \phi(0) + \phi(0) \Rightarrow \phi(0) = 0$

$\phi(1) = \phi(1 \cdot 1) = \phi(1) \cdot \phi(1)$ where $\phi(1) \neq \underbrace{0}_{\phi(0)}$ since ϕ is one-to-one. Multiply both sides by $\phi(1)^{-1}$ to get $\phi(1) = 1$.