Fields

Book II

Eg. $\alpha = \sqrt{2+\sqrt{2}}$

$\alpha^2 = 2+\sqrt{2}$

$\alpha^2 - 2 = \sqrt{2}$

$\alpha^4 - 4\alpha^2 + 4 = 2$

$\alpha^4 - 4\alpha^2 + 2 = 0$

The minimal poly. of α over $\mathbb{Q}$ is $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.

(Exercise: $f(x)$ is irreducible in $\mathbb{Q}[x]$ so it really is the min. poly. of α over $\mathbb{Q}$.)

The roots of $f(x)$ are

$\alpha = \sqrt{2+\sqrt{2}}$

$-\alpha = -\sqrt{2+\sqrt{2}}$

$\beta = \sqrt{2-\sqrt{2}}$

$-\beta = -\sqrt{2-\sqrt{2}}$

$$f(x) = x^4 - 4x^2 + 2 = (x-\alpha)(x+\alpha)(x-\beta)(x+\beta)$$

In this case $E = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 + d\alpha^3 : a,b,c,d \in \mathbb{Q}\}$ contains all the roots of $f(x)$ so it is a normal extension of $\mathbb{Q}$.

$\beta = (*) + (*)\alpha + (*)\alpha^2 + (*)\alpha^3 = \alpha^3 - 3\alpha$

$\alpha\beta = \sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}} = \sqrt{4-2} = \sqrt{2} = \alpha^2 - 2$

$\Rightarrow \beta = \dfrac{\alpha^2-2}{\alpha} \in \mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$

$\beta = \alpha - \frac{2}{\alpha} = \alpha - (4\alpha - \alpha^3) = \alpha^3 - 3\alpha$

$\alpha^4 - 4\alpha^2 + 2 = 0$

$\alpha^3 - 4\alpha + \frac{2}{\alpha} = 0 \Rightarrow \frac{2}{\alpha} = 4\alpha - \alpha^3$

$\alpha^4 = 4\alpha^2 - 2$

$\alpha^6 = 4\alpha^4 - 2\alpha^2$

$= 4(4\alpha^2-2) - 2\alpha^2$

$= 14\alpha^2 - 8$

Look for an automorphism $\sigma : E \to E$ $(E = \mathbb{Q}[\alpha])$ satisfying $\sigma(\alpha) = \beta$.

$\sigma(\beta) = \sigma(\alpha^3 - 3\alpha) = \sigma(\alpha)^3 - 3\sigma(\alpha) = \beta^3 - 3\beta = (\alpha^3-3\alpha)^3 - 3(\alpha^3 - 3\alpha) = (\alpha^3-3\alpha)((\alpha^3-3\alpha)^2 - 3)$

$= (\alpha^3-3\alpha)(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 3) = (\alpha^3-3\alpha)(14\alpha^2 - 8 - 6(4\alpha^2-2) + 9\alpha^2 - 3) = (\alpha^3-3\alpha)(-\alpha^2+1) = \alpha(\alpha^2-3)(-\alpha^2+1)$

$= \alpha(-\alpha^4 + 4\alpha^2 - 3) = \alpha(-(4\alpha^2-2) + 4\alpha^2 - 3) = -\alpha$

$\sigma : \alpha \longmapsto \beta = \alpha^3 - 3\alpha \longmapsto -\alpha \longmapsto -\beta \longmapsto \alpha$

$\mathrm{Aut}\, E = \langle\sigma\rangle$ of order 4 ; cyclic.

$\sigma(\sqrt{2}) = ?$

$\sqrt{2} = \alpha\beta$

$\sigma(\sqrt{2}) = \sigma(\alpha)\sigma(\beta)$

$= \beta\cdot(-\alpha)$

$= -\alpha\beta$

$= -\sqrt{2}$

$\sigma(\alpha) = \beta$

$\sigma(-\alpha) = -\sigma(\alpha) = -\beta$

$\sigma(\beta) = -\alpha$

$\sigma(-\beta) = -\sigma(\beta) = -(-\alpha)$

$= \alpha$

$\mathbb{Q}[\alpha]$

$2|$

$\mathbb{Q}[\sqrt{2}]$

$2|$

$\mathbb{Q}$

Galois correspondence

$G = \mathrm{Aut}\, E = \langle\sigma\rangle = \{1, \sigma, \sigma^2, \sigma^3\}$

$2|$

$\langle\sigma^2\rangle = \{1, \sigma^2\}$

$2|$

$\langle 1\rangle$

$\sigma^2(\sqrt{2}) = \sigma(\sigma(\sqrt{2}))$

$= \sigma(-\sqrt{2}) = \sqrt{2}$.

$\sigma(\sqrt{2}) = \sigma(\alpha^2 - 2)$

$= \sigma(\alpha)^2 - 2 = \beta^2 - 2 = -\sqrt{2}$.

Compare:

$$\begin{tikzcd}
E = \mathbb{Q}[\sqrt{2}] \arrow[d, dash, "2"'] \arrow[drrr, leftrightarrow] & & & G = \langle \tau \rangle \arrow[d, dash, "2"] \arrow[dlll, leftrightarrow] \\
\mathbb{Q} & & & \langle \iota \rangle
\end{tikzcd}$$

$G = \mathrm{Aut}\, E = \{\iota, \tau\}$, $\tau(a+b\sqrt{2}) = a - b\sqrt{2}$

Degree 2 extension : quadratic extension
" 3 " : cubic "
" 4 " : quartic "
" 5 " : quintic "

$\alpha = \sqrt[3]{2} = 2^{1/3}$

$E = \mathbb{Q}[\alpha] \supseteq \mathbb{Q}$ is an extension of degree $[E:\mathbb{Q}] = 3$ with basis $1, \underset{\sqrt[3]{2}}{\underset{\parallel}{\alpha}}, \alpha^2 = \sqrt[3]{4}$ $\quad (\alpha^3 = 2)$

α has min. poly. $x^3 - 2 \in \mathbb{Q}[x]$ which is irreducible

In $\mathbb{R}[x]$, $f(x) = x^3 - 2 = (x-\alpha)(x^2 + \alpha x + \alpha^2)$

Subfields

$$\begin{array}{c} E = \mathbb{Q}[\alpha] \\ |\,3 \\ \mathbb{Q} \end{array}$$

If $E \supseteq F \supseteq \mathbb{Q}$ (i.e. F is an intermediate field) then the transitivity of degrees tells us $\underbrace{[E:\mathbb{Q}]}_{3} = [E:F][F:\mathbb{Q}]$

$$3 \times 1 \quad \text{or} \quad 1 \times 3$$

If $[F:\mathbb{Q}] = 1$ then $\{1\}$ is a basis for F over $\mathbb{Q}$ so $F = \{a1 : a \in \mathbb{Q}\} = \mathbb{Q}$

If $[E:F] = 1$ then (similarly) $E = F$.

More generally if $E \supseteq F$ is an extension of prime degree $p = [E:F]$ then the only intermediate extensions are E and F.

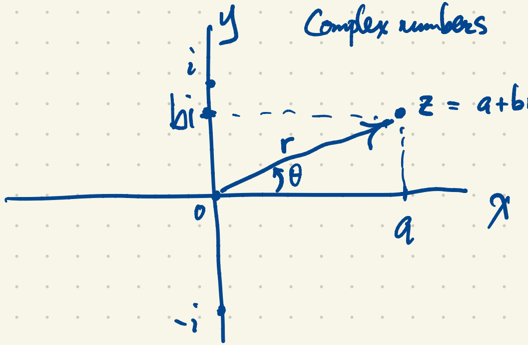
What are the automorphisms of $E = \mathbb{Q}[\alpha]$, $\alpha = \sqrt[3]{2}$? If $\phi \in \mathrm{Aut}\, E$ then $\phi(\alpha)^3 = \phi(\alpha^3) = \phi(2) = 2$

In $\mathbb{C}$, every poly. $f(x) \in \mathbb{C}[x]$ of degree n factors as $f(x) = a(x-r_1)(x-r_2)\cdots(x-r_n)$ $(a, r_1, r_2, \dots, r_n \in \mathbb{C})$.

eg. $x^n - 1 = (x-1)(x-\xi)(x-\xi^2)(x-\xi^3)\cdots(x-\xi^{n-1})$ where $\xi = e^{2\pi i/n}$.

de Moivre's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Complex numbers $\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$, $i = \sqrt{-1}$



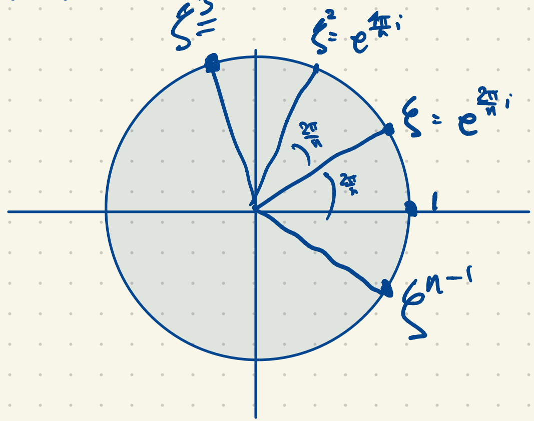
Every $z \in \mathbb{C}$ has unique representation as $z = a+bi$ $(a, b \in \mathbb{R})$ in rectangular coordinates

$a = \operatorname{Re} z$ = real part of z

$b = \operatorname{Im} z$ = imaginary part of z.

$r = |z| = \sqrt{a^2+b^2}$

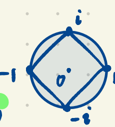
The roots of $x^n - 1$ are the n^{th} roots of unity: $1, \xi, \xi^2, \dots, \xi^{n-1}$ forming the vertices of a regular n-gon inscribed in the unit circle $|z| = 1$.



Eg. $n=4$

The fourth roots of unity are $\pm 1, \pm i$

$1, e^{\frac{i\pi}{2}} = i, e^{\frac{2i\pi}{2}} = e^{i\pi} = -1, e^{\frac{3i\pi}{2}} = -i$



Euler's Formula $e^{i\pi} = -1$

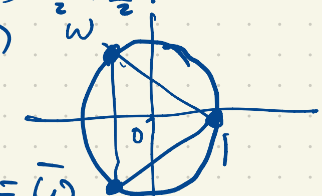
$e^{i\pi} + 1 = 0$

Eg. $n=3$: The three cube roots of unity in $\mathbb{C}$ are $1, \omega, \omega^2$ where

$\omega = e^{\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$

$\omega = \frac{-1 \pm \sqrt{-3}}{2}$



$\omega^2 = \overline{\omega}$

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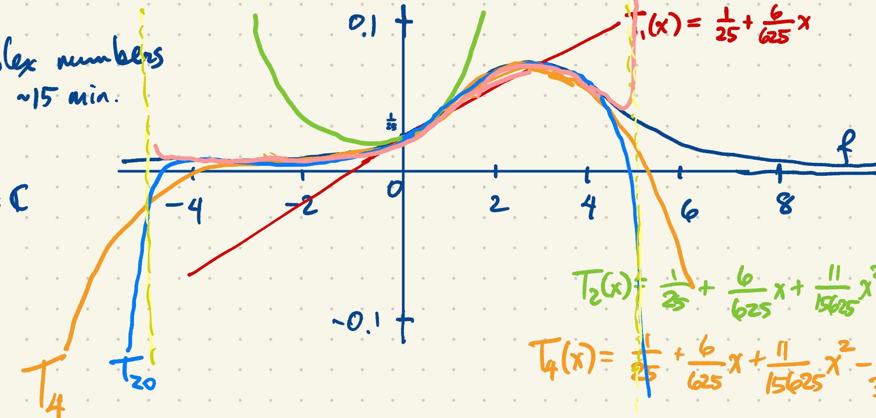
instructional videos $\longrightarrow$ complex numbers

~15 min.

Eg. consider $f(x) = \dfrac{1}{x^2-6x+25}$

This function has poles at $x = 3 \pm 4i \in \mathbb{C}$

with $|3 \pm 4i| = 5$



$T_1(x) = \frac{1}{25} + \frac{6}{625}x$

$T_2(x) = \frac{1}{25} + \frac{6}{625}x + \frac{11}{15625}x^2$

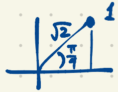
$T_4(x) = \frac{1}{25} + \frac{6}{625}x + \frac{11}{15625}x^2 - \frac{84}{390625}x^3 - \frac{779}{9765625}x^4$

By the Binomial Theorem

$(1+i)^{11} = 1 + 11i - 55 - 165i + 330 + 462i - 462 - 330i + 165 + 55i - 11 - i = -32 + 32i$

Much ~~faster~~ way to evaluate powers $z^n = (x+iy)^n = x^n + n x^{n-1} yi + \cdots + i^n y^n$ (Binomial Theorem)

$1+i = \sqrt{2} e^{i\frac{\pi}{4}}$

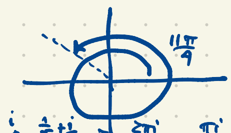


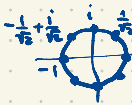
$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$(1+i)^{11} = \left(\sqrt{2} e^{\frac{i\pi}{4}}\right)^{11} = 32\sqrt{2}\, e^{\frac{i 11\pi}{4}}$

$= 32\sqrt{2} \cdot \left(\frac{-1+i}{\sqrt{2}}\right)$

$= -32 + 32i$





$= e^{\frac{2\pi i}{8}} = e^{\frac{\pi i}{4}}$ eighth root of unity

$\zeta = \frac{1+i}{\sqrt{2}}$

$\zeta^{11} = \zeta^3 = \frac{-1+i}{\sqrt{2}}$

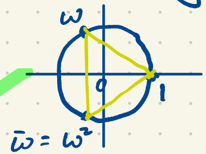
n^{th} roots of $z = re^{i\theta}$, $r = |z|$

all complex numbers whose n^{th} power is z

$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{i\theta}{n}},\ r^{\frac{1}{n}} e^{\frac{i\theta}{n} + \frac{2\pi i}{n}},\ r^{\frac{1}{n}} e^{\frac{i(\theta + 4\pi)}{n}},\ \ldots,\ r^{\frac{1}{n}} e^{i\frac{\theta + 2(n-1)\pi}{n}}$

i.e. $r^{\frac{1}{n}} e^{i\frac{\theta + 2k\pi}{n}}$, $k = 0, 1, 2, \ldots, n-1$

Cube roots of unity in $\mathbb{C}$: $1, \omega, \omega^2 = \bar{\omega}$



$\omega = e^{2\pi i/3} = \dfrac{-1+\sqrt{-3}}{2} = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2} i$

$\omega^2 + \omega + 1 = 0$

ω is a root of $x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$

$\tau(\omega) = \omega^2$

$$\begin{array}{ccc} \mathbb{Q}[\omega] & \longleftrightarrow & G = \langle \tau \rangle \\ 2\,| & & 2\,| \\ \mathbb{Q} & \longleftrightarrow & \langle \iota \rangle \end{array}$$

Galois correspondence

$G = \operatorname{Aut} \mathbb{Q}[\omega] = \langle \tau \rangle = \{\iota, \tau\}$
where $\tau(z) = \bar{z}$

Now let $\alpha = \sqrt[3]{2}$, $F = \mathbb{Q}[\alpha]$.

The min. poly. of α over $\mathbb{Q}$ is $x^3 - 2 \in \mathbb{Q}[x]$.

$F = \mathbb{Q}[\alpha] = \{ a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q} \}$, $\operatorname{Aut} F = \{\iota\}$

$3\,|$

$\mathbb{Q}$

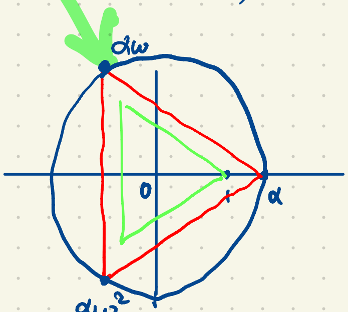
$x^3 - 2 = (x-\alpha)\underbrace{(x^2 + \alpha x + \alpha^2)}$

The other roots of $x^3 - 2$ are not in $F = \mathbb{Q}[\alpha]$ ie. the extension $F \supset \mathbb{Q}$ is not normal.

scale by factor of α

$x^3 - 2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ where $\alpha_1 = \alpha$, $\alpha_2 = \alpha\omega$, $\alpha_3 = \alpha\omega^2$

$x^3 - 2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$
$= (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$



$\alpha^3 = 2$

$(\alpha\omega)^3 = \alpha^3\omega^3 = 2 \cdot 1 = 2$

$(\alpha\omega^2)^3 = \alpha^3\omega^6 = 2 \cdot 1 = 2$

$E \supset F \supset \mathbb{Q}$

$E = \mathbb{Q}[\alpha_1, \alpha_2, \alpha_3] = \mathbb{Q}[\alpha, \omega]$, $F = \mathbb{Q}[\alpha]$

$\omega = \frac{1}{2}\alpha_1^2\alpha_2$
$= \frac{1}{2} \cdot 2^{2/3} \cdot 2^{1/3}\omega$
$= \omega$

There are $3! = 6$ permutations of $\alpha_1, \alpha_2, \alpha_3$.

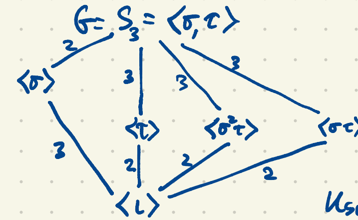
In $S_3 = \langle \sigma, \tau \rangle$, $\sigma = (123)$, $\tau = (23)$.

x	$\sigma(x)$	$\tau(x)$ ← complex conjugation
α_1	α_2	α_1
α_2	α_3	α_3
α_3	α_1	α_2
α	$\alpha\omega$	α
ω	ω	ω^2

$\sigma(\omega) = \sigma\left(\dfrac{\alpha_2}{\alpha_1}\right) = \dfrac{\sigma(\alpha_2)}{\sigma(\alpha_1)} = \dfrac{\alpha_3}{\alpha_2} = \dfrac{\alpha\omega^2}{\alpha\omega} = \omega$

$\sigma\left(\dfrac{\alpha\omega}{\alpha}\right)$

$\tau(\omega) = \tau\left(\dfrac{\alpha_2}{\alpha_1}\right) = \dfrac{\alpha_3}{\alpha_1} = \dfrac{\alpha\omega^2}{\alpha} = \omega^2 = \bar{\omega}$



$\sigma^2\tau = \tau\sigma$

$\sigma\tau = \tau\sigma^2$

Using right-to-left composition

$\sigma^2\tau = (132)(23) = (13)$

$\sigma\tau = (123)(23) = (12)$

$\tau = (23)$