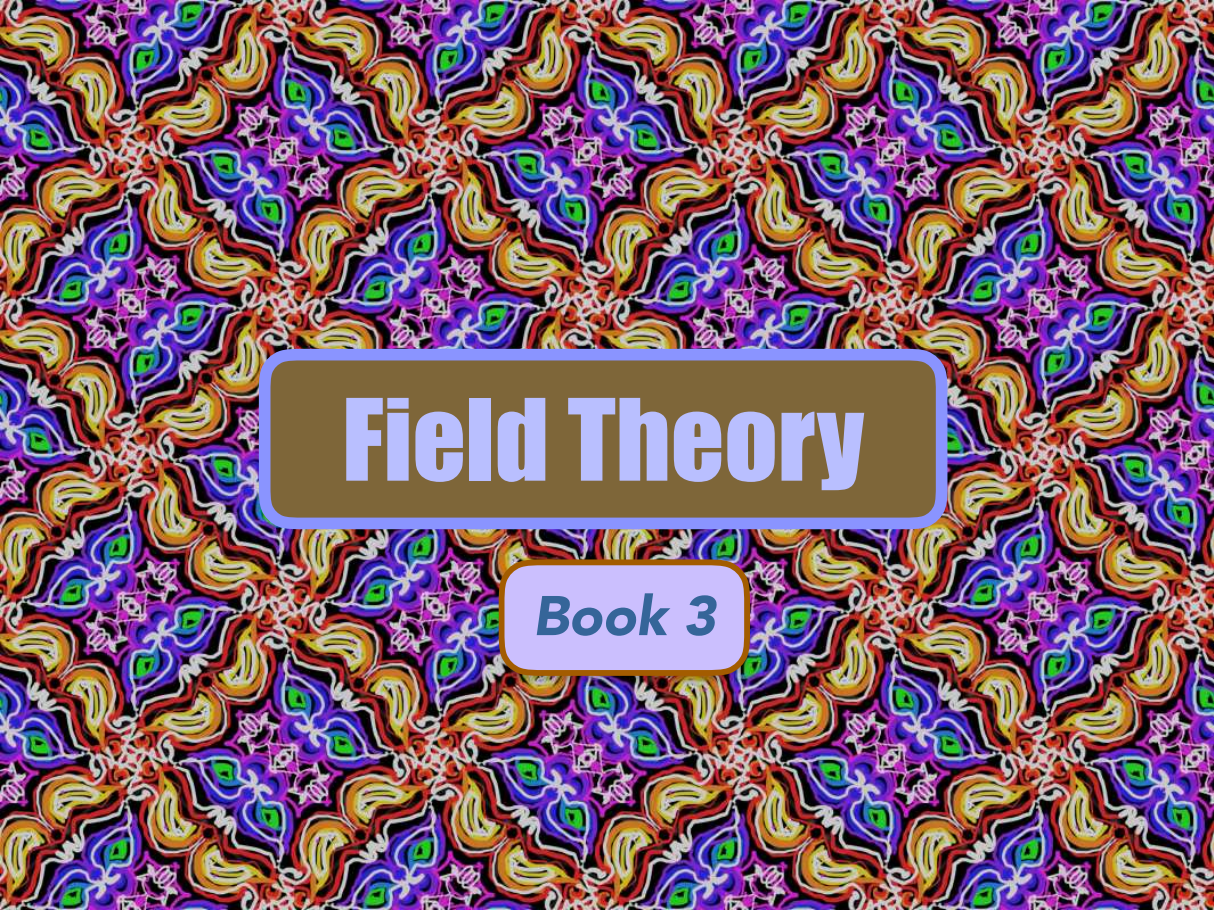




Field Theory

Book 3

Eg. $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$

x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

x	1^{2^0}	1^{2^1}	1^{2^2}	1^{2^3}
1	1	2	4	3
2	2	4	3	1
4	4	3	1	2
3	3	1	2	4

$\mathbb{F}_8$

x	1	θ	θ^2	θ^3	θ^4	θ^5	θ^6
1	1	θ	θ^2	θ^3	θ^4	θ^5	θ^6
θ	θ	θ^2	θ^3	θ^4	θ^5	θ^6	1
θ^2							
θ^3							
$\vdots$							

etc.

x	1	α	β	γ
1	1	α	β	γ
α	α	1	γ	β
β	β	γ	1	α
γ	γ	β	α	1

mult. table for a Klein 4-group
(noncyclic group of order 4)

This cannot be a subgroup in the multiplicative group of any field F for the following reason:

It has four solutions of $x^2 = 1$ (roots of $x^2 - 1$)

Wedderburn's Theorem: If F is any field (finite or infinite), then any subgroup of F^* (the multiplicative group of nonzero elements) is cyclic.

[If $F = \mathbb{F}_q$, then F^* is cyclic of order $q-1$.

The n^{th} roots of unity in $\mathbb{C}$ form a cyclic group of order n .

An extension $F \supseteq \mathbb{Q}$ (i.e. a field of characteristic zero) can be a finite extension or an infinite extension i.e.

- $n = [F:\mathbb{Q}] < \infty$: F is a finite extension of $\mathbb{Q}$ (i.e. an extension of finite degree n). These are number fields, also called algebraic number fields.

In this case F is a "simple" extension $F = \mathbb{Q}[\alpha] = \{a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_{n-1}\alpha^{n-1} : a_i \in \mathbb{Q}\}$.

eg. $F = \mathbb{Q}[\sqrt{2}, \sqrt{3}] = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} : a, b, c, d \in \mathbb{Q}\}$

so $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is a basis for F over $\mathbb{Q}$ and $[F:\mathbb{Q}] = 4$
(a quartic extension of $\mathbb{Q}$)

quadratic : degree 2
cubic : " 3
quartic : " 4
quintic : " 5

"Almost" any element $\alpha \in F$ generates F as a field: $F = \mathbb{Q}[\alpha]$.

If $\alpha = \sqrt{2} + \sqrt{3}$ then α has min. poly. $x^4 - 10x^2 + 1$

$$\alpha = \sqrt{2} + \sqrt{3}$$

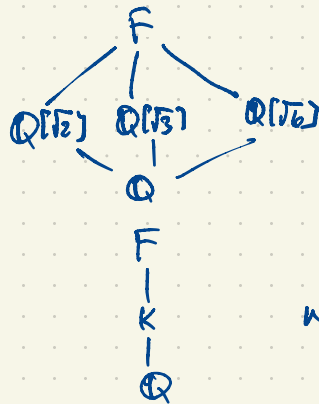
$$\alpha^2 = 2 + 3 + 2\sqrt{6} = 5 + 2\sqrt{6}$$

$$\alpha^2 - 5 = 2\sqrt{6}$$

$$\alpha^4 - 10\alpha^2 + 25 = 24$$

$$\alpha^4 - 10\alpha^2 + 1 = 0$$

All five subfields of F .



If $n = [F:\mathbb{Q}] < \infty$ then F has only finitely many subfields and all have degree dividing n

$$n = [F:\mathbb{Q}] = [F:K][K:\mathbb{Q}] \Rightarrow [K:\mathbb{Q}] \text{ divides } n.$$

Every element $\alpha \in F$ (if $n = [F:\mathbb{Q}] < \infty$)
is algebraic over $\mathbb{Q}$.

Why? If $\alpha \in F$, $[F:\mathbb{Q}] = n$, then $1, \alpha, \alpha^2, \dots, \alpha^n$ are linearly dependent over $\mathbb{Q}$.

$\Rightarrow a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_n\alpha^n = 0$ for some $a_0, a_1, \dots, a_n \in \mathbb{Q}$ not all zero.

$\Rightarrow \alpha$ is algebraic.

More than this, α is algebraic of degree dividing n .

$$F \supseteq \mathbb{Q}[\alpha] \supseteq \mathbb{Q} \Rightarrow n = [F:\mathbb{Q}] = [F:\mathbb{Q}[\alpha]] [\underbrace{\mathbb{Q}[\alpha]:\mathbb{Q}}_{\text{degree of } \alpha \text{ over } \mathbb{Q}}]$$

= degree of α over $\mathbb{Q}$

= degree of the min. poly.
of α over $\mathbb{Q}$.

• $[F:\mathbb{Q}] = \infty$ eg. $\mathbb{R}, \mathbb{C}, \dots$

Let $\alpha \in \mathbb{R}$ or $\mathbb{C}$ and consider the field $\mathbb{Q}(\alpha)$ generated by α . This is the smallest extension of $\mathbb{Q}$ containing α . If α is algebraic, this gives a finite extension $\mathbb{Q}[\alpha]$, $n = [\mathbb{Q}(\alpha):\mathbb{Q}] < \infty$.

Let's take π which is known to be transcendental. $\mathbb{Q}(\pi)$ is the subfield of $\mathbb{R}$ containing π . $\mathbb{Q} \subset \mathbb{Q}(\pi) \subset \mathbb{R}$.

The subring $\mathbb{Q}[\pi] \subset \mathbb{R}$ generated by π using addition, subtraction, and multiplication only, is the subring

$$\mathbb{Q}[\pi] = \{a_0 + a_1\pi + a_2\pi^2 + \dots + a_n\pi^n : a_0, a_1, \dots, a_n \in \mathbb{Q}, n \geq 0\}$$

eg. $\frac{28}{5}\pi^4 - \frac{19}{11}\pi^3 + 13\pi^2 + 105\pi - \frac{13}{11} \in \mathbb{Q}[\pi]$. It's not a field: $\pi \in \mathbb{Q}[\pi]$, $\frac{1}{\pi} \notin \mathbb{Q}[\pi]$.

If $\frac{1}{\pi} = a_0 + a_1\pi + a_2\pi^2 + \dots + a_n\pi^n$ for some n , and $a_0, a_1, \dots, a_n \in \mathbb{Q}$ then

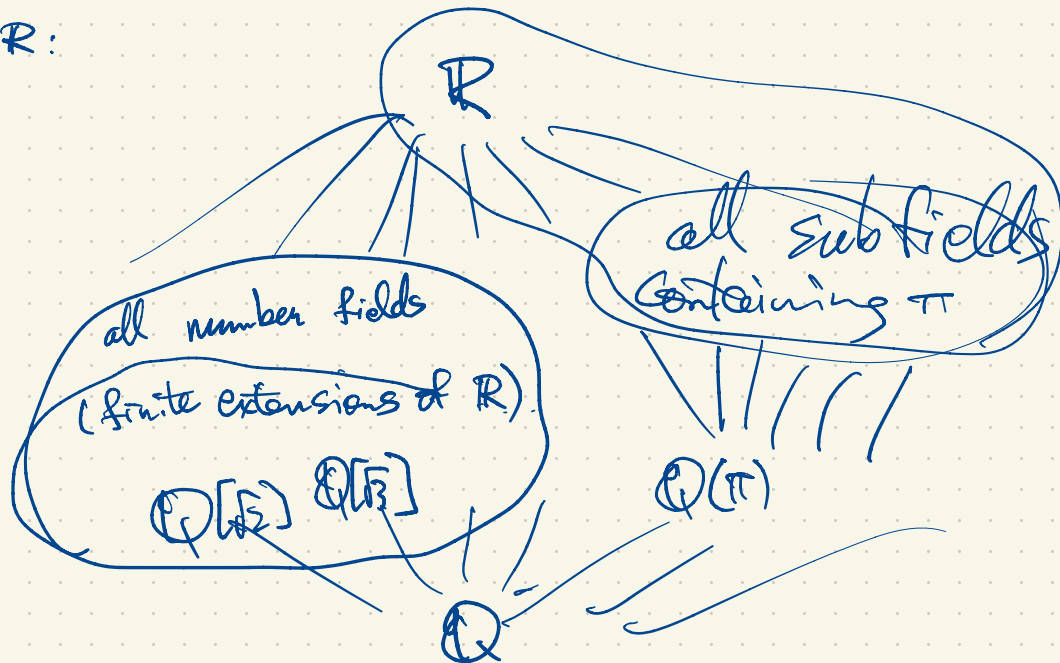
$$a_n\pi^{n+1} + a_{n-1}\pi^n + \dots + a_2\pi^3 + a_1\pi^2 + a_0\pi - 1 = 0. \quad \text{Contradiction.}$$

To extend the ring $\mathbb{Q}[\pi]$ to a field, we divide elements of $\mathbb{Q}[\pi]$ inside $\mathbb{R}$:

$$\mathbb{Q}(\pi) = \left\{ \frac{f(\pi)}{g(\pi)} : f(x), g(x) \in \mathbb{Q}[x], \quad g(x) \neq 0 \right\}.$$

This is a field. It's a subfield. It's generated by π under $+$, $-$, $\times$, $\div$
So it is the smallest subfield containing π .

Subfields of $\mathbb{R}$:



About notation: $F[x]$ = the ring of all polynomials in x ^(symbol/indeterminate) with coefficients in F .

$F(x)$ = the field of all rational functions in x with coefficients in F .
 $= \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in F[x], g(x) \neq 0 \right\}$.

$F(x)$ is the field of quotients of $F[x]$.

$\mathbb{Q} \dots \dots \dots \mathbb{Z}$.

$$\begin{aligned}\mathbb{Q}[\sqrt{2}] &= \{ f(\sqrt{2}) : f(x) \in \mathbb{Q}[x] \} \\ &= \{ a + b\sqrt{2} : a, b \in \mathbb{Q} \}.\end{aligned}$$

$$\mathbb{Q}(\sqrt{2}) = \left\{ \frac{u}{v} : u, v \in \mathbb{Q}[\sqrt{2}], v \neq 0 \right\} = \mathbb{Q}[\sqrt{2}]$$

(This ring $\mathbb{Q}[\sqrt{2}]$ is already a field)

$$\begin{aligned}\text{If } f(x) &= \frac{5}{3}x^3 + x^2 - 4x + \frac{11}{2} \in \mathbb{Q}[x] \\ \text{then } f(\sqrt{2}) &= \frac{5}{3}\sqrt{2}^3 + \sqrt{2}^2 - 4\sqrt{2} + \frac{11}{2} \\ &= \frac{10}{3}\sqrt{2} + 2 - 4\sqrt{2} + \frac{11}{2} \\ &= \frac{15}{2} - \frac{2}{3}\sqrt{2}\end{aligned}$$

$$\mathbb{Q}(\pi) \supset \mathbb{Q}[\pi].$$

ring which
is not a field

Let $\alpha \in \mathbb{C}$. Then $\mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$ iff α is algebraic.

there are infinitely many quadratic extensions of $\mathbb{Q}$. Suppose $E \supset \mathbb{Q}$ with $[E:\mathbb{Q}] = 2$.
Then E has a basis $\{1, \theta\}$ i.e. every element of E is uniquely expressible as
 $a + b\theta$ $\frac{a, b \in \mathbb{Q}}$. Since $\theta^2 \in E$, $\theta^2 = a + b\theta$ for some $a, b \in \mathbb{Q}$, i.e. $\theta^2 - b\theta - a = 0$.
 $\Rightarrow \theta = \frac{b \pm \sqrt{b^2 + 4a}}{2} = \frac{b \pm \sqrt{\delta}}{2}$ where $\delta = b^2 + 4a \in \mathbb{Q}$. Now $E = \mathbb{Q}[\theta] = \mathbb{Q}[\sqrt{\delta}]$

$$\text{eg. } \theta^2 = \frac{5}{2}\theta + 3 \Rightarrow \theta^2 - \frac{5}{2}\theta - 3 = 0 \Rightarrow \theta = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} + 12}}{2} = \frac{\frac{5}{2} \pm \frac{1}{2}\sqrt{25+48}}{2} = \frac{5 \pm \sqrt{73}}{4}$$

$$\mathbb{Q}\left[\frac{5+\sqrt{73}}{4}\right] = \mathbb{Q}\left[\frac{5-\sqrt{73}}{4}\right] = \mathbb{Q}[\sqrt{73}]$$

$$\mathbb{Q}[\sqrt{12}] = \mathbb{Q}[6\sqrt{2}] = \mathbb{Q}[\sqrt{2}]$$

Above: a quadratic extension $E \supset \mathbb{Q}$, $[E:\mathbb{Q}] = 2$. If $\alpha \in E - \mathbb{Q}$ then $\mathbb{Q}[\alpha] = E$.

$$E \supset \mathbb{Q}[\alpha] \supset \mathbb{Q} \Rightarrow 2 = [E:\mathbb{Q}] = \underbrace{[E:\mathbb{Q}[\alpha]]}_{=1} \underbrace{[\mathbb{Q}[\alpha]:\mathbb{Q}]}_{2 \neq 1} \Rightarrow$$

$$\mathbb{Q}[\theta] = \mathbb{Q}[\sqrt{73}] \text{ where } \theta \text{ is any root of } x^2 - \frac{5}{2}x - 3.$$

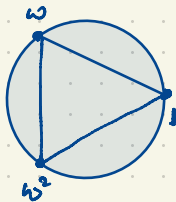
All the quadratic extensions of $\mathbb{Q}$ are

$$\sqrt{8} = 2\sqrt{2} \Rightarrow \mathbb{Q}[\sqrt{8}] = \mathbb{Q}[\sqrt{2}]$$

..., $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{13}]$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt{17}]$, $\mathbb{Q}[\sqrt{26}]$, $\mathbb{Q}[\sqrt{10}]$, $\mathbb{Q}[\sqrt{14}]$, ...

No two fields in this list are the same. In fact, no two of them are isomorphic.

If ω is a root of $x^2 + x + 1$ then $\omega^3 = 1$ and so ω is a cube root of unity.



$$\underbrace{x^3 - 1}_{\text{roots: } 1, \omega, \omega^2} = (x-1) \underbrace{(x^2 + x + 1)}_{\text{root: } 1} \quad \underbrace{\omega, \omega^2}_{\text{roots: } \omega, \omega^2 \text{ (primitive cube roots of 1)}}$$

$$[\mathbb{Q}[\omega]:\mathbb{Q}] = 2$$

So it's in our list above... which one is it?

$$\mathbb{Q}[\omega] = \mathbb{Q}[\sqrt{-3}] \text{ Same field!}$$

$$x^2 + x + 1 \text{ has roots } \frac{-1 \pm \sqrt{-3}}{2} = \omega, \omega^2.$$

$$\omega = \frac{-1 + \sqrt{-3}}{2} \Rightarrow \sqrt{-3} = 2\omega + 1$$

Clarify what we mean by: the same field.

If $K, K' \subseteq \mathbb{C}$ are two subfields then one of three things can happen:

- $K = K'$
- or • $K \neq K'$ but $K \cong K'$ (isomorphic)
- or • $K \not\cong K'$ (not isomorphic).

Eg. $\mathbb{Q}[\omega] = \mathbb{Q}[\sqrt{-3}]$ (two subfields of $\mathbb{C}$ that are actually equal)

$\mathbb{Q}[\sqrt{2}] \neq \mathbb{Q}[\sqrt{3}]$. The field $\mathbb{Q}[\sqrt{2}]$ has roots of $x^2 - 2$ but $\mathbb{Q}[\sqrt{3}]$ has no roots of $x^2 - 2$ ($x^2 - 2$ is reducible in one field but irreducible in the other.)

An isomorphism between two fields $\phi: K \rightarrow K'$ is a bijection such that $\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta)$ and $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ for any $\alpha, \beta \in K$.
for any isomorphism $\phi: K \rightarrow K'$, we must have $\phi(0) = 0$.
Proof: $\phi(0) = \phi(0+0) = \phi(0) + \phi(0)$. Subtract $\phi(0)$ from both sides to get $\phi(0) = 0$.
Next: show $\phi(1) = 1$ in K' . Proof: $\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1)$ where $\phi(1) \neq 0$. Since ϕ is bijective, multiply both sides by $\phi(1)^{-1}$ so $\phi(1)^{-1}\phi(1) = \underbrace{\phi(1)^{-1}\phi(1)}_1 \phi(1) = \phi(1)$.

$$2 = 1 + 1$$

$$\phi(2) = \phi(1+1) = \phi(1) + \phi(1) = 1 + 1 = 2$$

$$3 = 1 + 1 + 1 = 2 + 1$$

$$\phi(3) = \phi(2+1) = \phi(2) + \phi(1) = 2 + 1 = 3$$

... $\phi(n) = n$ for every positive integer

$\phi(a) = a$ for all $a \in \mathbb{Q}$, $n = \underbrace{1+1+\dots+1}_{n \text{ terms}}$

$$\text{char } K = \text{char } K'$$

If $\text{char } K = 0$ then the prime subfield of K is $\mathbb{Q}$. Same for K' .

In this case $K \supseteq \mathbb{Q}$ and $K' \supseteq \mathbb{Q}$ and $\phi(a) = a$ for all $a \in \mathbb{Q}$.

Suppose $\phi: \mathbb{Q}[\sqrt{2}] \rightarrow \mathbb{Q}[\sqrt{3}]$ is an isomorphism. Then $\phi(\sqrt{2}) = a + b\sqrt{3}$, $a, b \in \mathbb{Q}$.

$$\text{Then } \phi(\sqrt{2})^2 = (a + b\sqrt{3})^2 \quad (a + b\sqrt{3})^2 = 2$$

$$= \phi(\sqrt{2})\phi(\sqrt{2})$$

$$2 + 3b^2 + 2ab\sqrt{3} = 2$$

$$2ab\sqrt{3} = 2 - a^2 - 3b^2$$

$$\phi(\sqrt{2}\sqrt{2}) = \phi(2) = 2$$

If $ab \neq 0$ then $\sqrt{3} = \frac{2 - a^2 - 3b^2}{2ab} \in \mathbb{Q}$
contradicting Euclid's proof.

If $b = 0$ then $a^2 = 2$

$a = \pm\sqrt{2} \in \mathbb{Q}$
contradiction

If $a = 0$ then

$$3b^2 = 2$$

$$9b^2 = 6$$

$$(3b)^2 = 6$$

$$\sqrt{6} = \pm 3b \in \mathbb{Q}$$

contradiction
by Euclid

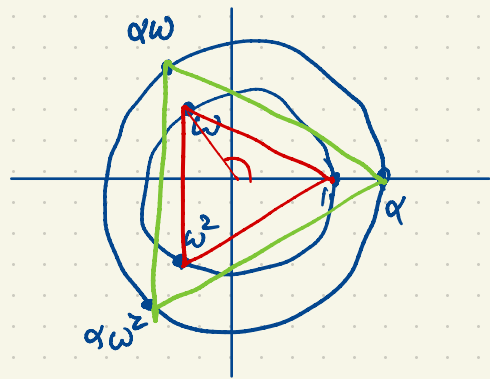
Third possibility: $K, K' \subseteq \mathbb{C}$ subfields
 $K \neq K'$ but $K \cong K'$.

To give an example we need an extension of degree ≥ 3 .

$m(x) = x^3 - 2 \in \mathbb{Q}[x]$ is irreducible.

n^{th} roots of any nonzero complex number form vertices of a regular n -gon in $\mathbb{C}$

$m(x)$ has roots $\alpha = 2^{1/3}$ (the unique real root)



$1, \omega, \omega^2$ are the three cube roots of unity

$$x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)$$

$$\omega = e^{2\pi i/3}$$

ω, ω^2 are the two primitive cube roots of 1.

$$\alpha^3 = 2$$

$$(\alpha\omega)^3 = \alpha^3 \omega^3 = 2 \cdot 1 = 2$$

$$(\alpha\omega^2)^3 = \alpha^3 \omega^6 = 2 \cdot 1 = 2$$

$\left. \begin{array}{l} \alpha_1 = \alpha \\ \alpha_2 = \alpha\omega \\ \alpha_3 = \alpha\omega^2 \end{array} \right\}$ three roots of $m(x)$

$$m(x) = x^3 - 2 = (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$$

$$\mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\} \subset \mathbb{R}$$

$$[\mathbb{Q}[\alpha] : \mathbb{Q}] = 3. \quad \text{with } 1, \alpha, \alpha^2 \text{ as basis.}$$

$$\mathbb{Q}[\alpha_2] = \{a + b\alpha_2 + c\alpha_2^2 : a, b, c \in \mathbb{Q}\} \subset \mathbb{C}$$

$$\underline{\mathbb{Q}[\alpha_2] \not\subset \mathbb{R}}$$

$$[\mathbb{Q}[\alpha_2] : \mathbb{Q}] = 3. \quad \text{with basis } 1, \alpha_2, \alpha_2^2$$

$$\left. \begin{array}{l} \mathbb{Q}[\alpha_1] = \mathbb{Q}[\alpha] \\ \mathbb{Q}[\alpha_2] = \mathbb{Q}[\alpha\omega] \end{array} \right\} \text{ Not equal. because } \mathbb{Q}[\alpha] \subset \mathbb{R} \text{ but } \mathbb{Q}[\alpha_2] \not\subset \mathbb{R}.$$

$$\text{But } \mathbb{Q}[\alpha_1] \cong \mathbb{Q}[\alpha_2]$$

$$\text{The isomorphism } \phi: \mathbb{Q}[\alpha_1] \longrightarrow \mathbb{Q}[\alpha_2] \text{ is}$$

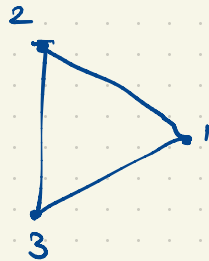
$$\phi(a + b\alpha_1 + c\alpha_1^2) = a + b\alpha_2 + c\alpha_2^2$$

eg. in $\mathbb{Q}[\alpha]$, $(1+\alpha+2\alpha^2)(3-\alpha) = 3+2\alpha+5\alpha^2-2\frac{\alpha^3}{2} = -1+2\alpha+5\alpha^2$
 in $\mathbb{Q}[\alpha_2]$, $(1+\alpha_2+2\alpha_2^2)(3-\alpha_2) = -1+2\alpha_2+5\alpha_2^2$

The three roots of $m(x) = x^3-2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) = (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$ has three roots that are completely interchangeable.

$S_3 = \{\text{all permutations of } 1, 2, 3\}$, $|S_3| = 3! = 6$.

$S_3 \cong$ symmetry group of an equilateral triangle



In $\mathbb{Q}[\sqrt{2}]$, we have an isomorphism $\phi: \mathbb{Q}[\sqrt{2}] \rightarrow \mathbb{Q}[\sqrt{2}]$

An automorphism of a field is an isomorphism from the field to itself.

	1	ϕ
$\sqrt{2}$	1	ϕ
ϕ	ϕ	1

$\phi(a+b\sqrt{2}) = a-b\sqrt{2}$

is an isomorphism.

i.e. ϕ is bijective

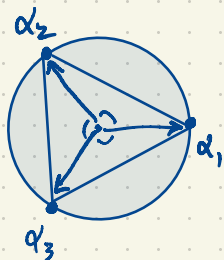
The automorphisms of $\mathbb{Q}[\sqrt{2}]$ form a (cyclic) group $\{1, \phi\}$ of order 2

$\phi(uv) = \phi(u)\phi(v)$

$\phi(u+v) = \phi(u) + \phi(v)$

for all u, v .

$x^3 - 2$ has three complex roots $\alpha_1, \alpha_2, \alpha_3$. Are $\alpha_1, \alpha_2, \alpha_3$ linearly dependent or not over $\mathbb{Q}$?



$\alpha_1 + \alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_1, \alpha_2, \alpha_3$ are linearly dependent over $\mathbb{Q}$.

You can't have three lin. indep. vectors in $\mathbb{C}$ over $\mathbb{R}$ because $[\mathbb{C} : \mathbb{R}] = 2$.

$\alpha_1, \alpha_2, \alpha_3$ lie in an extension $E = \mathbb{Q}[\alpha_1, \alpha_2, \alpha_3]$

If $m(x) = x^3 + bx + c$, $b, c \in \mathbb{Q}$ then the roots of $m(x)$ in $\mathbb{C}$ are linearly dependent over $\mathbb{Q}$
 $= (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$, some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ not necessarily distinct.
 $= x^3 - (\alpha_1 + \alpha_2 + \alpha_3)x^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)x - \alpha_1\alpha_2\alpha_3$

$$\Leftrightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = b \\ \alpha_1\alpha_2\alpha_3 = -c \end{cases}$$

Eg. $E = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$ has four automorphisms $1, \sigma, \tau, \sigma\tau$

$$= \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} : a, b, c, d \in \mathbb{Q}\}$$

$[E : \mathbb{Q}] = 4$ since $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ is a basis

$$1(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}) = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$$

$$\sigma(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}) = a - b\sqrt{2} + c\sqrt{3} - d\sqrt{6}$$

$$\tau(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}) = a + b\sqrt{2} - c\sqrt{3} - d\sqrt{6}$$

$$\sigma\tau(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}) = a - b\sqrt{2} - c\sqrt{3} + d\sqrt{6}$$

These are the only automorphisms of E

θ	$1(\theta)$	$\sigma(\theta)$	$\tau(\theta)$	$\sigma\tau(\theta)$
1	1	1	1	1
$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$
$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$
$\sqrt{6}$	$\sqrt{6}$	$-\sqrt{6}$	$-\sqrt{6}$	$\sqrt{6}$

$$1: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\tau: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\sigma\tau: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

	1	σ	τ	$\sigma\tau$
1	1	σ	τ	$\sigma\tau$
σ	σ	1	$\sigma\tau$	τ
τ	τ	$\sigma\tau$	1	σ
$\sigma\tau$	$\sigma\tau$	τ	σ	1

Klein
four-group

$$\begin{aligned} \sigma\tau &= \tau\sigma \\ \sigma^2 &= 1 \\ \tau^2 &= 1 \end{aligned}$$

Every field has at least one automorphism

$$1(\alpha) = \alpha \text{ for all } \alpha$$

$$\begin{aligned} \sigma(\sqrt{6}) &= \sigma(\sqrt{2}\sqrt{3}) = \sigma(\sqrt{2})\sigma(\sqrt{3}) \\ &= (-\sqrt{2})(\sqrt{3}) = -\sqrt{6} \end{aligned}$$

$$\begin{aligned} \tau(\sqrt{6}) &= \tau(\sqrt{2})\tau(\sqrt{3}) \\ &= \sqrt{2}(-\sqrt{3}) = -\sqrt{6} \end{aligned}$$

$$\begin{aligned} \sigma\tau(\sqrt{6}) &= \sigma\tau(\sqrt{2})\sigma\tau(\sqrt{3}) \\ &= (-\sqrt{2})(-\sqrt{3}) \\ &= \sqrt{6} \end{aligned}$$

If $E \supseteq \mathbb{Q}$ is an extension of degree $n = [E:\mathbb{Q}] < \infty$ (number field)
then E has at most n automorphisms.

when $n=2$, then E has exactly 2 automorphisms.

Consider $E = \mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\alpha]$ where $\alpha = \sqrt{2} + \sqrt{3}$.

α has min. poly. $m(x) = x^4 - 10x^2 + 1$ over $\mathbb{Q}$.

$$= (x-\alpha)(x+\alpha)(x-\beta)(x+\beta)$$

$$\beta = \sqrt{2} - \sqrt{3}$$

Roots of $m(x)$:

- $\alpha = \sqrt{2} + \sqrt{3}$
- $\beta = \sqrt{2} - \sqrt{3}$
- $-\beta = -\sqrt{2} + \sqrt{3}$
- $-\alpha = -\sqrt{2} - \sqrt{3}$

Suppose ϕ is any automorphism of E . Then $\phi(a) = a$ for all $a \in \mathbb{Q}$.

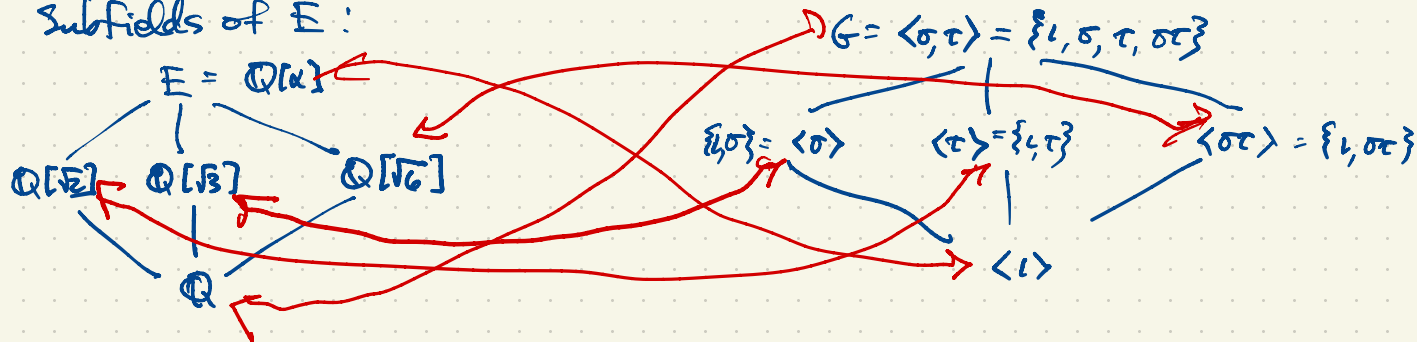
$$m(\phi(\alpha)) = \phi(\alpha)^4 - 10\phi(\alpha)^2 + 1 = \phi(\alpha^4 - 10\alpha^2 + 1) = \phi(0) = 0$$

$\Rightarrow \phi(\alpha) \in \{\alpha, -\alpha, \beta, -\beta\} \Rightarrow \phi = \text{id}, \sigma\tau, \tau, \sigma$ respectively

Since ϕ is determined by its action on a basis

As above $E = \mathbb{Q}[\alpha, \beta]$, $\alpha = \sqrt{2} + \sqrt{3}$, $\beta = \sqrt{2} - \sqrt{3}$ containing $\frac{1}{2}(\alpha + \beta) = \sqrt{2}$ and $\frac{1}{2}(\alpha - \beta) = \sqrt{3}$
 $= \mathbb{Q}[\sqrt{2}, \sqrt{3}]$.

Subfields of E :



$G = \text{Aut } E = \{ \text{automorphisms of the field } E \}$ is a group.

If $[E:\mathbb{Q}] = n$ then $|G| \leq n$.

What is $\text{Aut } \mathbb{Q}$? $\text{Aut } \mathbb{R}$? $\text{Aut } \mathbb{C}$?

If $\phi: \mathbb{Q} \rightarrow \mathbb{Q}$ is an isomorphism (i.e. an automorphism of $\mathbb{Q}$) then $\phi(a) = a$ for all $a \in \mathbb{Q}$ as explained before, i.e. $\phi = 1$ = identity function. $\text{Aut } \mathbb{Q} = \{1\}$.

If $\phi: \mathbb{C} \rightarrow \mathbb{C}$ is an automorphism then $\phi = 1$ or τ or ...

$$1(z) = z$$

$$\tau(z) = \bar{z} \quad \text{where} \quad \tau(a+bi) = a-bi \quad \text{for all } a, b \in \mathbb{R}$$

$$\tau \text{ is bijective since } \tau^{-1} = \tau \quad \text{i.e.} \quad \tau^2 = 1 \quad \tau(\tau(z)) = \bar{\bar{z}} = z$$

$$\tau(z + z') = \overline{z + z'} = \bar{z} + \bar{z}' = \tau(z) + \tau(z')$$

$$z, z' \in \mathbb{C}$$

$$\tau(zz') = \tau(z)\tau(z')$$

$$\overline{zz'} = \bar{z}\bar{z}'$$

$1, \tau \in \text{Aut } \mathbb{C}$ Are there other automorphisms?

Galois correspondence

subgroups of $G = \text{Aut } E$ and
Subfields of E .

Fact: If $E \supseteq \mathbb{Q}$ is a number field i.e. $n = [E:\mathbb{Q}] < \infty$ then E has at most n automorphisms. Why? later...

If $E \supseteq \mathbb{Q}$, $[E:\mathbb{Q}] = n$ with exactly n automorphisms (the maximum number) then the extension $E \supseteq \mathbb{Q}$ is called Galois. In this case there is a one-to-one correspondence between

Eg. $E = \mathbb{Q}[\alpha]$ where α is a root of $f(x) = x^3 - 3x + 1$.
 $f(x)$ is irreducible in $\mathbb{Q}[x]$,
 $f(x) = (x-\alpha)(x-\beta)(x-\gamma)$

$$[E:\mathbb{Q}] = 3.$$

We will show that E has 3 automorphisms.

$$G = \text{Aut } E = \{1, \sigma, \sigma^2\}, \quad \sigma^3 = 1$$

G is cyclic of order 3.

We'll see that $\beta = \alpha^2 - 2$ is also a root of $f(x)$. $\beta \in \mathbb{Q}[\alpha]$ since $\beta = \alpha^2 - 2$

$$f(\beta) = \beta^3 - 3\beta + 1$$

$$= (\alpha^2 - 2)^3 - 3(\alpha^2 - 2) + 1$$

$$= (\alpha^6 - 6\alpha^4 + 12\alpha^2 - 8) - (3\alpha^2 - 6) + 1$$

$$= \alpha^6 - 6\alpha^4 + 9\alpha^2 - 1$$

$$= (9\alpha^2 - 6\alpha + 1) - 6(3\alpha^2 - \alpha) + 9\alpha^2 - 1$$

$$= 0$$

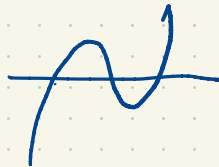
But can $\beta = \alpha$ (same root)? We must have $\beta \neq \alpha$
 Since $1, \alpha, \alpha^2$ are lin. indep.

(Or: if $\alpha = \beta = \alpha^2 - 2$ then $\alpha^2 - \alpha - 2 = 0$, a contradiction
 since the minimal poly. of α .)

Note that $\gamma = \beta^2 - 2$ must also be a root of $f(x)$.

Also $\delta = \gamma^2 - 2$ is also a root.

Same argument as before.



$E = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$
 $1, \alpha, \alpha^2$ is a basis for E
 over $\mathbb{Q}$

$$\alpha^3 - 3\alpha + 1 = 0 \Rightarrow \alpha^3 = 3\alpha - 1$$

$$\alpha^4 = 3\alpha^2 - \alpha$$

$$\alpha^5 = 3\alpha^3 - \alpha^2 = 3(3\alpha - 1) - \alpha^2$$

$$= -\alpha^2 + 9\alpha - 3$$

$$\alpha^6 = -\alpha^3 + 9\alpha^2 - 3\alpha$$

$$= -(3\alpha - 1) + 9\alpha^2 - 3\alpha$$

$$= 9\alpha^2 - 6\alpha + 1$$

Since $\gamma = \beta^2 - 2$
 since $\alpha = \gamma^2 - 2$

Alternatively

$$\alpha^6 = (\alpha^3)^2 = (3\alpha - 1)^2$$

$$= 9\alpha^2 - 6\alpha + 1$$

$$\gamma = \beta^2 - 2 = (\alpha^2 - 2)^2 - 2$$

$$= \alpha^4 - 4\alpha^2 + 4 - 2$$

$$= \alpha^4 - 4\alpha^2 + 2$$

$$= (3\alpha^2 - \alpha) - 4\alpha^2 + 2$$

$$= -\alpha^2 - \alpha + 2$$

$$\delta = \gamma^2 - 2$$

$$= (\alpha^2 - \alpha + 2)^2 - 2 = \alpha^4 + \alpha^2 + 4 + 2\alpha^3 - 4\alpha^2 - 4\alpha - 2$$

$$= \alpha^4 + 2\alpha^3 - 3\alpha^2 - 4\alpha + 2$$

$$= (3\alpha^2 - \alpha) + 2(3\alpha - 1) - 3\alpha^2 - 4\alpha + 2$$

$$= \alpha$$

In the example above, $E = \mathbb{Q}[\alpha] = \mathbb{Q}[\beta] = \mathbb{Q}[\gamma]$ has exactly three automorphisms.

$$G = \text{Aut } E = \{1, \sigma, \sigma^2\}, \quad \sigma: \alpha \mapsto \beta \mapsto \gamma \mapsto \alpha. \quad (3\text{-cycle}) \quad \text{like } (1,2,3) \in S_3.$$

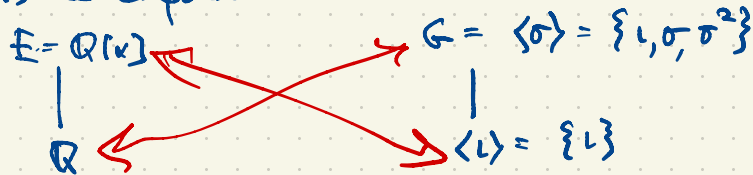
Given $u = a + b\alpha + c\alpha^2 \in E$, $a, b, c \in \mathbb{Q}$, $\sigma(u) = \sigma(a + b\alpha + c\alpha^2) = \underbrace{\sigma(a)}_a + \underbrace{\sigma(b\alpha)}_{\substack{\sigma(b)\sigma(\alpha) \\ = b\sigma(\alpha) \\ = b\beta}} + \underbrace{\sigma(c\alpha^2)}_{\substack{\sigma(c)\sigma(\alpha)^2 \\ = c\beta^2}} = a + b\beta + c\beta^2$

$$\sigma^2(u) = a + b\gamma + c\gamma^2$$

$$\sigma^3(u) = u, \quad \sigma^3 = 1$$

$E \supset \mathbb{Q}$ is a Galois extension with $|G| = 3 = [E:\mathbb{Q}]$.

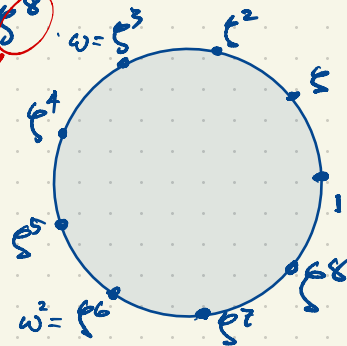
Galois correspondence between subfields of E and subgroups of G :



Where does the map $\alpha \mapsto \alpha^2 - 2$ come from? Look at $\mathbb{Q}[\zeta]$, ζ is a primitive 9th root of unity. In $\mathbb{C}$, the roots of $x^9 - 1$ are $1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5, \zeta^6, \zeta^7, \zeta^8$.

$1, \zeta^3, \zeta^6$ imprimitive 9th roots of 1.
min. poly. $x^3 - 1$
primitive cube roots
 $x^2 + x + 1$

Six primitive 9th roots of 1.
roots of $x^6 + x^3 + 1$



$$x^9 - 1 = \underbrace{(x-1)}_{\text{root } 1} \underbrace{(x^2 + x + 1)}_{\text{roots } \zeta^3, \zeta^6} \underbrace{(x^6 + x^3 + 1)}_{\text{roots } \zeta, \zeta^2, \zeta^4, \zeta^5, \zeta^7, \zeta^8}$$

$$F = \mathbb{Q}[\xi] = \{a_0 + a_1\xi + a_2\xi^2 + \dots + a_5\xi^5 : a_0, a_1, \dots, a_5 \in \mathbb{Q}\} \supset \mathbb{Q}$$

extension of degree 6 with basis $1, \xi, \xi^2, \xi^3, \xi^4, \xi^5$

min. poly. of ξ over $\mathbb{Q}$ is $x^6 + x^3 + 1$.

This extension has six automorphisms $\Gamma = \text{Aut } F = \{1, \phi, \phi^2, \phi^3, \phi^4, \phi^5\}$

$$\phi(\xi) = \xi^2$$

$$\phi(\xi^2) = \phi(\xi)\phi(\xi) = \xi^2\xi^2 = \xi^4$$

$$\phi(\xi^3) = \phi(\xi)^3 = (\xi^2)^3 = \xi^6 = -1 - \xi^3$$

$$\phi(\xi^4) = \phi(\xi^2)^2 = (\xi^4)^2 = \xi^8 = \xi^{-1} = -\xi^2 - \xi^5$$

$$\begin{aligned} \phi(\omega) &= \omega^2 = -1 - \omega \\ \phi^2(\omega) &= \phi(-1 - \omega) = -1 - (-1 - \omega) = \omega \end{aligned}$$

$$\begin{aligned} \xi^7 &= 1 \\ \xi^8 &= \xi^{-1} = \bar{\xi} \end{aligned}$$

$$\xi^2 + \xi^5 + \xi^8 = 0$$

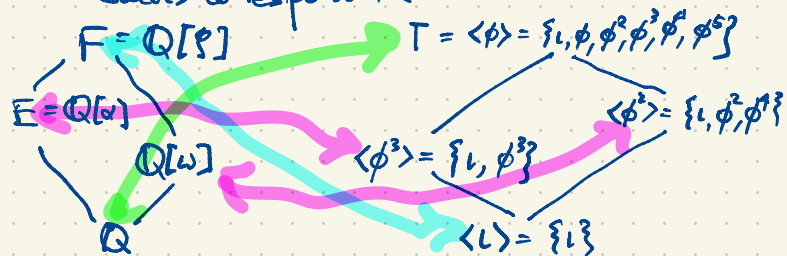
$$\phi(\xi^8) = \xi^{16} = \xi^7\xi^9 = \xi^7$$

$$\phi(\xi^7) = \xi^{14} = \xi^8\xi^5 = \xi^5$$

$$\phi(\xi^5) = \xi^{10} = \xi^8\xi^2 = \xi$$

$\phi: \xi \mapsto \xi^2 \mapsto \xi^4 \mapsto \xi^8 \mapsto \xi^7 \mapsto \xi^5 \mapsto \xi$
 ϕ cycles the six primitive 9th roots of unity in a 6-cycle. like $(123456) \in S_6$

Galois Correspondence



$$\begin{aligned} \alpha &= \xi + \xi^8 = \xi + \xi^{-1} \\ \phi(\alpha) &= \phi(\xi) + \phi(\xi^8) = \xi^2 + \xi^7 \\ \phi^2(\alpha) &= \phi(\phi(\alpha)) = \phi(\xi^2) + \phi(\xi^7) = \xi^4 + \xi^5 \\ \phi^3(\alpha) &= \phi(\xi^4 + \xi^5) = \xi^8 + \xi = \alpha \\ \alpha^2 &= (\xi + \xi^{-1})^2 = \xi^2 + \xi^{-2} + 2 \Rightarrow \beta = \xi^2 + \xi^{-2} = \alpha^2 - 2 \end{aligned}$$

New example: $f(x) = x^3 - 2$ irreducible over $\mathbb{Q}$.
Let α be a root of $f(x)$. This generates an extension

$$F = \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\} \supset \mathbb{Q}$$
$$[f: \mathbb{Q}] = 3 \quad \text{with basis} \quad 1, \alpha, \alpha^2$$

$\text{Aut } F = \{1\}$. Only one automorphism of F .

This is not a Galois extension of $\mathbb{Q}$.

If $\phi \in \text{Aut } F$ (an automorphism of F)

then what is $\phi(x)$?

$$\phi(\alpha)^3 = \phi(\alpha^3) = \phi(2) = 2 \Rightarrow \phi(\alpha) = \alpha \Rightarrow \phi = 1$$

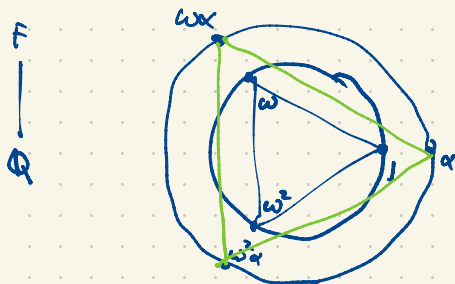
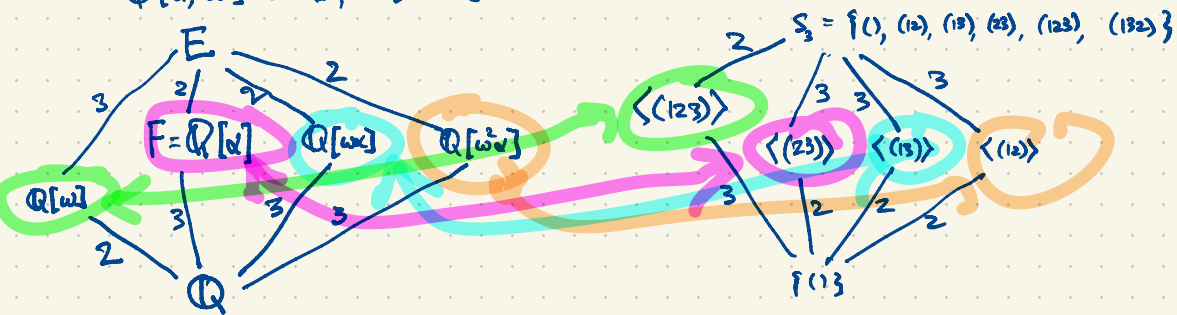
We're missing the other cube roots of 2. $f(x) = x^3 - 2 = (x - \alpha)(x^2 + \alpha x + \alpha^2) = (x - \alpha)(x - \omega\alpha)(x - \omega^2\alpha)$

$$\alpha = 2^{1/3} \quad (\text{real cube root of } 2)$$

$$\alpha^3 = 2$$
$$(\omega\alpha)^3 = \omega^3 \alpha^3 = 1 \cdot 2 = 2$$

$E = \mathbb{Q}[\alpha, \omega\alpha, \omega^2\alpha] =$ splitting field of $f(x) = x^3 - 2$. Splitting fields are Galois extensions all (here we throw in the roots of $f(x)$).

$$= \mathbb{Q}[\alpha, \omega] \supset \mathbb{Q}, \quad [E:\mathbb{Q}] = 6 \quad G = \text{Aut } E \text{ has order } |G| = 6$$



$$\omega = e^{2\pi i/3} = \frac{-1 + i\sqrt{3}}{2}$$

ω has min. poly. x^2+x+1
 $\omega^2+\omega+1=0$

$$E = \mathbb{Q}[\alpha, \omega]$$

$$\alpha = 2^{1/3}, \quad \omega^3 + \omega + 1 = 0$$

S_3 permutes $\alpha_1, \alpha_2, \alpha_3$ the three roots of $f(x) = x^3 - 2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$

$\tau = (23)$ maps $\alpha_1 \mapsto \alpha_1, \quad \alpha_2 \leftrightarrow \alpha_3$

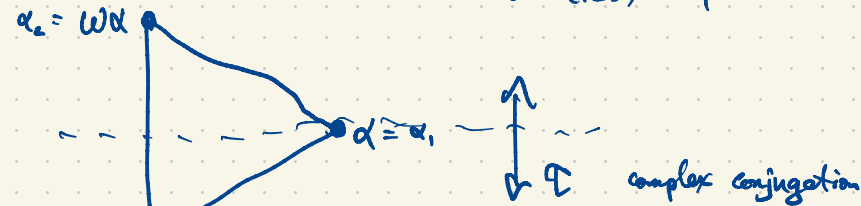
$\sigma = (123)$ maps $\alpha_1 \mapsto \alpha_2 \mapsto \alpha_3 \mapsto \alpha_1$

$$\sigma(\alpha) = \omega\alpha$$

$$\sigma(\omega\alpha) = \omega^2\alpha$$

$$\sigma(\omega^2\alpha) = \alpha$$

$$\sigma(\omega) = \sigma\left(\frac{\omega\alpha}{\alpha}\right) = \frac{\sigma(\omega\alpha)}{\sigma(\alpha)} = \frac{\omega^2\alpha}{\omega\alpha} = \omega$$

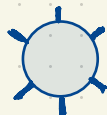


$$G = \langle \sigma, \tau \rangle = \left\{ \underset{\substack{(1) \\ (123)}}{1}, \underset{\substack{(132)}}{\sigma}, \underset{\substack{(123)}}{\sigma^2}, \underset{\substack{(23)}}{\tau}, \sigma\tau, \sigma^2\tau \right\}$$

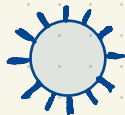
$\mathbb{Q} \subset \mathbb{Q}[\sqrt{2}] \subset \dots \subset \mathbb{R}$

two automorphisms

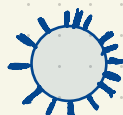
no nontrivial automorphisms



12 symmetries



24 symmetries



1 symmetry

If $\phi \in \text{Aut } \mathbb{R}$ i.e. ϕ is an automorphism of $\mathbb{R}$ i.e. $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is bijective, $\phi(x+y) = \phi(x) + \phi(y)$, $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}$, then $\phi = 1$ i.e. $\phi(x) = x$ for all $x \in \mathbb{R}$. Why?

As before, $\phi(x) = x$ for all $x \in \mathbb{R}$.

Can $\phi(\sqrt{2}) = -\sqrt{2}$? No. If $\phi(\sqrt{2}) = -\sqrt{2}$ then consider $\alpha = 2^{1/4}$ so $\alpha^2 = \sqrt{2}$, so $\phi(\alpha)^2 = \phi(\alpha)\phi(\alpha) = \phi(\alpha^2) = \phi(\sqrt{2}) = -\sqrt{2}$. There is no possible value $\phi(\alpha) \in \mathbb{R}$ such that $\phi(\alpha)^2 < 0$. Contradiction.

$\phi(\sqrt{2}) = \sqrt{2}$. In fact $\phi(x) = x$ for all $x \in \mathbb{R}$.

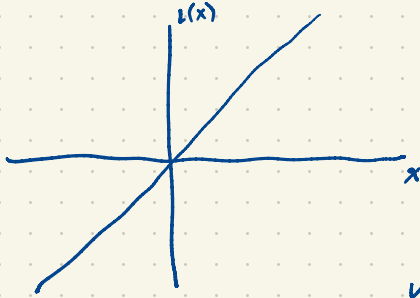
How many automorphisms does the field $\mathbb{C}$ have? $1, \tau$ ($\tau(z) = \bar{z}$)
 $1(\bar{z}) = z$

In fact $\text{Aut } \mathbb{C}$ is huge.

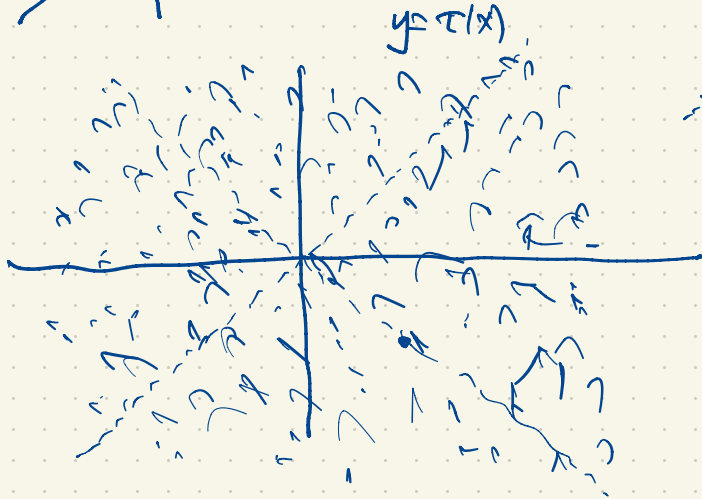
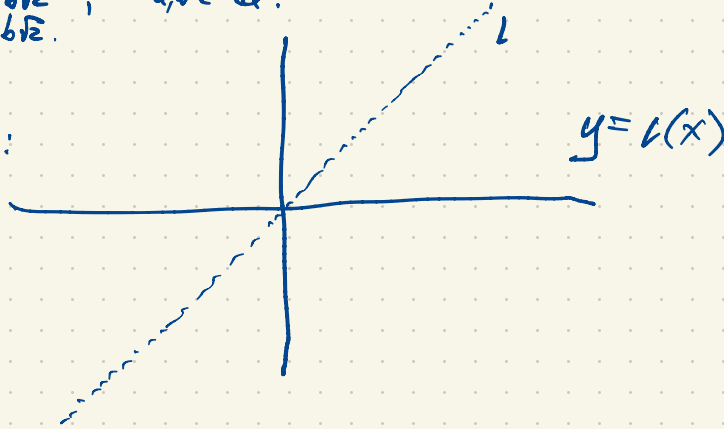
Most automorphisms of $\mathbb{C}$ are discontinuous everywhere.

The only continuous automorphisms of $\mathbb{C}$ are $1, \tau$.

$\text{Aut } \mathbb{Q}[\sqrt{2}] = \{1, \tau\}$, $\tau(a+b\sqrt{2}) = a-b\sqrt{2}$, $a, b \in \mathbb{Q}$.
 $1(a+b\sqrt{2}) = a+b\sqrt{2}$.



Actually:



Suppose $f(x) \in \mathbb{Q}[x]$ is a nonzero polynomial (without loss of generality, f is monic).

If $f(x)$ is irreducible, then the roots of $f(x)$ ($\in \mathbb{C}$) are distinct.

Supposing $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = (x-r_1)(x-r_2)\dots(x-r_n)$, $r_1, r_2, \dots, r_n \in \mathbb{C}$,
irreducible in $\mathbb{Q}[x]$

Then $r_1, \dots, r_n$ are distinct.

More generally if $\text{char } F = 0$ and $f(x) \in F[x]$ is irreducible,
then the roots of $f(x)$ (in any extension field) are distinct.