Math 4520

Fields

Book 1

A *field* is a number system with $+, -, \times, \div, 0, 1$ in which we can divide by every nonzero element.

$\mathbb{R} = \{\text{real numbers}\}$ is a field

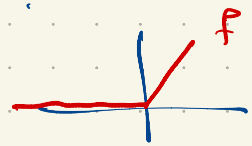
$\mathbb{C} = \{\text{complex numbers}\}$ " " $\qquad \mathbb{C} = \{a+bi : a,b \in \mathbb{R}\}, \quad i = \sqrt{-1}$

$\mathbb{Q} = \{\text{rational numbers}\}$ " " $\qquad \mathbb{Q} = \{\frac{a}{b} : a,b \in \mathbb{Z}, b \neq 0\}$

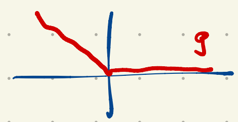
$\mathbb{Z} = \{\text{integers}\} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$ is not a field. $\mathbb{Z}$ is a commutative ring with 1 (having no zero divisors)

Other examples of fields?

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$



$$g(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



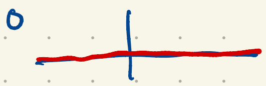
$f(x) + g(x) = |x|$

$f(x) - g(x) = x$

$f(x)g(x) = \underline{0}$

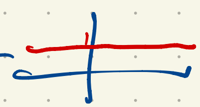
$\cancel{\frac{f(x)}{g(x)}}$

$f \neq 0$



$f(x)$ and $g(x)$ are nonzero functions whose product is 0 (these are called <u>zero divisors</u> in ring theory).

The continuous functions $\mathbb{R} \to \mathbb{R}$ form a commutative ring with identity but not a field.



$\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ is a field of <u>order</u> 5.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$3 + 4 = 2$

$3 - 4 = 4$

$3 \times 4 = 2$

$\frac{3}{4} = 2$

$\mathbb{R}, \mathbb{C}, \mathbb{Q}$ are infinite fields; $\mathbb{F}_5$ is a finite field.

In $\mathbb{Z}$, $8 \equiv 3 \bmod 5$. (8 is congruent to 3 mod 5)

In $\mathbb{F}_5$, $8 = 3$.

$$\frac{3}{4} = 2 \iff 3 = 2 \times 4$$

$2 \times 4 = 8 = 3$

Fields

Let F be a set containing distinct elements called 0 and 1 (thus $0 \neq 1$). Suppose addition, subtraction, multiplication and division are defined for all elements of F (except division by 0 is not defined).
Thus $a+b,\ a-b,\ ab,\ \frac{a}{d} \in F$ whenever $a, b, d \in F$ and $d \neq 0$.
Define $-a = 0 - a$.

If the following properties are satisfied by *all* elements $a, b, c, d \in F$ with $d \neq 0$, then F is a field.

$$a + b = b + a \qquad a + (b + c) = (a + b) + c \qquad ab = ba$$

$$a + 0 = a \qquad a(bc) = (ab)c \qquad 1a = a$$

$$a + (-a) = 0 \qquad a(b + c) = ab + ac \qquad \frac{a}{d} d = a$$

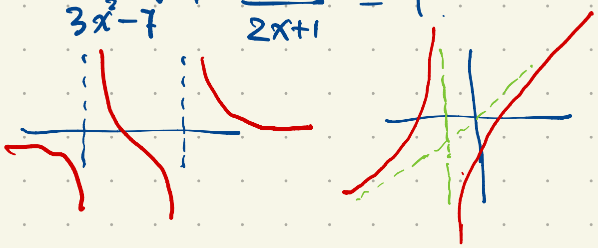
$$a + (-b) = a - b$$

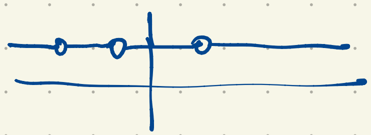
The ring of polynomials in x with real coefficients is $\mathbb{R}[x]$ is a commutative ring with identity having no zero divisors (if $f(x), g(x)$ are nonzero polynomials then $f(x)g(x)$ is a nonzero polynomial)

$\underline{1}\, f(x) = f(x)$

$\mathbb{R}(x) = \{$ rational functions in x with real coefficients $\}$, a field.

$$\frac{2x+1}{3x^2-7} \cdot \frac{3x^2-7}{2x+1} = 1$$





$$x \cdot \frac{1}{x} = 1$$

$$\mathbb{R}(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{R}[x], \quad g(x) \neq 0 \right\}.$$

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, \quad b \neq 0 \right\}.$$

$\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} : a, b \in \mathbb{Q} \}$ is a field.

$$\left.\begin{aligned} \alpha &= 3 + 7\sqrt{2} \\ \beta &= 5 - \sqrt{2} \end{aligned}\right\} \in \mathbb{Q}[\sqrt{2}]$$

$\alpha + \beta = (3 + 7\sqrt{2}) + (5 - \sqrt{2}) = 8 + 6\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

$\alpha - \beta = (3 + 7\sqrt{2}) - (5 - \sqrt{2}) = -2 + 8\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

$\alpha\beta = (3 + 7\sqrt{2})(5 - \sqrt{2}) = 15 - 3\sqrt{2} + 35\sqrt{2} - 14 = 1 + 32\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

$$\frac{\alpha}{\beta} = \frac{3 + 7\sqrt{2}}{5 - \sqrt{2}} \cdot \frac{5 + \sqrt{2}}{5 + \sqrt{2}} = \frac{15 + 3\sqrt{2} + 35\sqrt{2} + 14}{25 - 2} = \frac{29 + 38\sqrt{2}}{23} = \frac{29}{23} + \frac{38}{23}\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

$\mathbb{Q}[i] = \{ a + bi : a, b \in \mathbb{Q} \}$, $i = \sqrt{-1}$ This is a field.

$\alpha = 1 + \frac{1}{3}i$, $\beta = \frac{3}{2} - i$ $\Rightarrow$ $\alpha + \beta = \frac{5}{2} - \frac{2}{3}i$

$\alpha - \beta = -\frac{1}{2} + \frac{4}{3}i$

$\alpha\beta = \frac{3}{2} - i + \frac{1}{2}i + \frac{1}{3} = \frac{11}{6} - \frac{1}{2}i \in \mathbb{Q}[i]$

$$\frac{\alpha}{\beta} = \frac{1 + \frac{1}{3}i}{\frac{3}{2} - i} \cdot \frac{6}{6} = \frac{6 + 2i}{9 - 6i} \cdot \frac{9 + 6i}{9 + 6i} = \frac{54 + 36i + 18i - 12}{81 + 36}$$

$$= \frac{42 + 54i}{117} = \frac{14}{39} + \frac{18}{39}i \in \mathbb{Q}[i].$$