UNIVERSITY OF WYOMING

Math 4520—Fall 2024

Algebra III

Fields

Department of Mathematics

Sample Test

November, 2024

This sample test is intended to resemble the upcoming Test (Monday, November 4) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of material covered in class and on our handouts.

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an $8.5'' \times 11''$ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Total value of questions: 100 points (plus 8 bonus points).

1. *(30 points)* Let $\zeta = e^{i\pi/3}$. Note that the complex primitive sixth roots of unity are ζ and ζ^5.

   (a) Determine $m(x)$, the minimal polynomial of ζ over $\mathbb{Q}$.

   (b) Let $E = \mathbb{Q}[\zeta]$. Determine the degree $[E:\mathbb{Q}]$ of this extension, and find an explicit basis for E over Q.

   (c) Find all elements of $G = \mathsf{Aut}\, E$. What is the order $|G|$? Is G cyclic? Is G abelian?

   (d) Show that $\sqrt{-3} \in E$.

   (e) Determine the two Hasse diagrams (for the subfields of E, and for the subgroups of G), thereby illustrating the Galois correspondence.

2. *(15 points)* Let $E \supseteq \mathbb{Q}$ be a field extension of degree $[E:\mathbb{Q}] = n$. Assume that $G = \mathsf{Aut}\, E$ has order n (equivalently, the extension is normal and hence Galois; it is the splitting field of a polynomial of degree n). For each $\alpha \in E$, the *norm* and the *trace* of α are defined by

$$N(\alpha) = \prod_{g\in G} g(\alpha) \qquad T(\alpha) = \sum_{g\in G} g(\alpha),$$

   these being the product and the sum (respectively) of all images of α under the action of G. Show that $N(\alpha)$ and $T(\alpha)$ are rational numbers.

   (*Hint*: Show that $N(\alpha)$ and $T(\alpha)$ are fixed by every element of G; that is, $\tau(N(\alpha)) = N(\alpha)$ and $\tau(T(\alpha)) = T(\alpha)$. So the Galois correspondence applies.)

3. *(15 points)* Let $\alpha \in \mathbb{C}$ be algebraic with minimal polynomial $f(x) = x^3+3x^2+x+1 \in \mathbb{Q}[x]$. (You may use the fact that $f(x)$ is irreducible in $\mathbb{Q}[x]$.) Determine the minimal polynomial of $2\alpha+1$ (in simplified form).

4. *(18 points)* Fill in the blanks using words selected from the following list:

subring	field	subfield	inverse	extension	conjugate
value	unit	negative	reducible	homomorphism	one-to-one
nonzero	zero	degree	dimension	commutative	associative

to make the following a valid proof. (*Not all words* in the list will be used.)

Theorem. *If σ is an automorphism of a field F, then the subset $K \subseteq F$ defined by $K = \{a \in F : \sigma(a) = a\}$ is a subfield of F.*

Proof. Since $\sigma(0) = \sigma(0+0) = \sigma(0) + \sigma(0)$, we obtain $\sigma(0) = 0$, so $0 \in K$. Also since $\sigma(1) = \sigma(1{\cdot}1) = \sigma(1)\sigma(1)$, we have $\sigma(1) \in \{0, 1\}$; however $\sigma(1) \neq 0$ since σ is ______, so $\sigma(1) = 1$ and $1 \in K$.

Now suppose $a, b \in K$. Then $\sigma(a+b) = \sigma(a) + \sigma(b) = a+b$, so $a+b \in K$; also $\sigma(ab) = \sigma(a)\sigma(b) = ab$, so $ab \in K$. Furthermore, $0 = \sigma(0) = \sigma(a + (-a)) = \sigma(a) + \sigma(-a) = a + \sigma(-a)$, so $\sigma(-a) = -a$. Thus $-a \in K$, whence K is a ______ of F.

Finally, suppose $a \in K$ is ______; we must show that a is a ______ in K. Since F is a ______, there exists an ______ $a' \in F$ satisfying $aa' = a'a = 1$; we must show that $a' \in K$. Since $1 = \sigma(1) = \sigma(aa') = \sigma(a)\sigma(a') = a\sigma(a')$ and inverses are unique in F, we must have $\sigma(a') = a'$ so $a' \in K$. $\square$

5. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) Every subfield of $\mathbb{R}$ is infinite. ______*(True/False)*

(b) The only extension field $E \supseteq \mathbb{C}$ is $E = \mathbb{C}$ itself. ______*(True/False)*

(c) If F and F' are subfields of a field E, then $F \cap F'$ is also a subfield of E. ______*(True/False)*

(d) Every field extension $E \supseteq \mathbb{Q}$ of degree $[E : \mathbb{Q}] = n$ has n automorphisms. ______*(True/False)*

(e) If $F, F' \subset \mathbb{C}$ are subfields with both $[F : \mathbb{Q}]$ and $[F' : \mathbb{Q}]$ finite, then there exists a finite extension field $E \supseteq \mathbb{Q}$ containing both F and F'. ______*(True/False)*

(f) If $\alpha \in \mathbb{C}$ satisfies $\alpha^2 - \alpha - 3 = 0$, then $\mathbb{Q}[\alpha] = \mathbb{Q}[\sqrt{13}\,]$. ______*(True/False)*

(g) If $E \supseteq \mathbb{Q}$ is a finite extension field, then $E = \mathbb{Q}[\alpha]$ for some $\alpha \in E$.

______*(True/False)*

(h) If S is the set of all invertible 2×2 matrices with rational entries, then $S \cup \left\{ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \right\}$ is a field. ______*(True/False)*

(i) For every positive integer n, there is at least one extension $E \supseteq \mathbb{Q}$ of degree $[E : \mathbb{Q}] = n$. ______*(True/False)*

(j) If σ is an automorphism of a field extension $E \supseteq \mathbb{Q}$ and $z \mapsto \overline{z}$ is complex conjugation, then $\sigma(\overline{z}) = \overline{\sigma(z)}$ for all $z \in E$. ______*(True/False)*