University of Wyoming
Department of Mathematics

Math 4520—Fall 2024

Algebra III

Fields

Practice Problems 1

October, 2024

The following are problems suggested for further practice on the content prior to the Test on Monday, November 4. These problems range in difficulty, from easy to challenging (yet do-able).

1. Find the minimal polynomial of $\alpha = 2^{1/3} + 2^{2/3}$ over $\mathbb{Q}$.

2. Let ζ be a complex root of $f(x) = x^4 + 1 \in \mathbb{Q}[x]$. Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$, and that $f(x) = (x-\zeta)(x-\zeta^3)(x-\zeta^5)(x-\zeta^7)$. Find all automorphisms of the field $E = \mathbb{Q}[\zeta]$.

3. Imitate one of the examples done in class to show that the splitting field $E \supset \mathbb{Q}$ of the polynomial $f(x) = x^3 - 3x + 1$ is a Galois extension with cyclic automorphism group $G = \operatorname{Aut} E$. (*Hint:* Check that the map $t \mapsto 2 - t - t^2$ cyclically permutes the three roots of $f(x)$.)

4. Let $E \supset \mathbb{Q}$ be a quadratic extension, i.e. a field extension of degree 2. Show that $E \cong \mathbb{Q}[\sqrt{m}]$ where m is an integer, $m \equiv 1, 2$ or $3 \mod 4$.

5. Let $\alpha = \sqrt{5} + \sqrt{22 + 2\sqrt{5}}$.
   (a) Find the minimal polynomial of α over $\mathbb{Q}$.
   (b) Prove that $\sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}} \;=\; \alpha.$

6. Let $\alpha, \beta, \gamma \in \mathbb{C}$ be the three roots of $m(x) = x^3 - 7x^2 + 5x + 3$. Compute each of the following. Each of your answers should be expressed simply as an integer.
   (a) $\alpha + \beta + \gamma$
   (b) $\alpha\beta\gamma$
   (c) $\alpha^2 + \beta^2 + \gamma^2$

7. Find three 3×3 matrices A, B, C with rational entries satisfying

$$A + B + C = -I \qquad AB + AC + BC = -2I, \qquad ABC = I,$$
$$A^2 - 2I = B, \qquad B^2 - 2I = C, \qquad C^2 - 2I = A.$$

Theorem 1. *There exist irrational numbers α and β such that α^β is rational.*

Proof. Let $\gamma = \sqrt{2}^{\sqrt{2}}$. If γ is rational then we are done. Otherwise $\gamma \notin \mathbb{Q}$ and $\gamma^{\sqrt{2}} = \sqrt{2}^2 = 2$ is rational as required. $\square$

The proof given for Theorem 1 is *nonconstructive* in the sense that it proves the assertion without actually providing an example. (It does not actually answer whether or not $\sqrt{2}^{\sqrt{2}}$ is rational). Recall that a number $\alpha \in \mathbb{C}$ is *algebraic* (over $\mathbb{Q}$) if it is a root of some nonzero polynomial $f(x) \in \mathbb{Q}[x]$. In this case, $f(x)$ can be taken to be monic (its leading coefficient is 1); and the *minimal* polynomial of α (over $\mathbb{Q}$) is the unique monic polynomial $f(x) \in \mathbb{Q}[x]$ of smallest possible degree, having α as a root. In this case, $f(x) \in \mathbb{Q}[x]$ is irreducible; and the polynomials in $\mathbb{Q}[x]$ having α as a root are precisely the multiples of $f(x)$. If α is *not* algebraic over $\mathbb{Q}$, then we say it is transcendental.

The *Gelfond-Schneider Theorem* asserts that if α, β are complex numbers, algebraic over $\mathbb{Q}$, with $\alpha \notin \{0, 1\}$ and β irrational, then every value of α^β is transcendental.

'Every value' is a reminder that α^β may have more than one value, just as $\alpha^{1/2}$ has two values whenever $\alpha \neq 0$. To define α^β, first write $\alpha = e^\gamma$ and then take $\alpha^\beta = e^{\beta\gamma}$. Although the exponential function $z \mapsto e^z$ is well-defined, the value of $\gamma = \ln \alpha$ is not well-defined: If $\alpha = re^{i\theta}$ where $r, \theta \in \mathbb{R}$ with $r > 0$, then we may take $\gamma = \ln r + (\theta + 2k\pi)i$ where k an arbitrary integer; and then $\alpha^\beta = e^{\beta \ln r + \beta(\theta + 2k\pi)i}$ is another possible value for α^β.

8. (a) Find all possible values of i^i. (Use the fact that $e^{x+yi} = e^x e^{yi} = e^x(\cos y + i \sin y)$ whenever $x, y \in \mathbb{R}$.) By the Gelfond-Schneider Theorem, all these values of i^i are transcendental.

   (b) Use the Gelfond-Schneider Theorem to give a constructive proof of Theorem 1 above.

9. Using the fact that π is transcendental, show that $\sqrt{\pi^2 - 1}$ is also transcendental.

In preparation for #10,11, let us evaluate the continued fraction $\alpha = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}}$ in closed form. Since $\alpha = 1 + \frac{1}{\alpha}$, we see that $\alpha^2 - \alpha - 1 = 0$ and so $\alpha = \frac{1 \pm \sqrt{5}}{2}$. Evidently $\alpha > 0$, and so $\alpha = \frac{1+\sqrt{5}}{2}$.

10. Is $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\text{etc.}}}}}$ rational, algebraic irrational, or transcendental? Justify your answer.

11. Let $x = \sqrt{5 + \sqrt{5 - \sqrt{5 + \sqrt{5 - \sqrt{5 + \sqrt{5 - \cdots}}}}}}$. Show that x is algebraic, and find its minimal polynomial over $\mathbb{Q}$.

12. Let $\alpha = 2^{1/3}$. Show that α cannot be expressed as a rational linear combination of square roots of rational numbers, i.e. for any rational numbers a_1, b_i $(i = 1, 2, \ldots, n)$, the value of $a_1\sqrt{b_1} + a_2\sqrt{b_2} + \cdots + a_n\sqrt{b_n}$ cannot equal α.