Solutions to HW2

2. The fifth roots of $x^5-1=(x-1)m(x)$ in $\mathbb{C}$ are $1,\zeta,\zeta^2,\zeta^3,\zeta^4$. Since 1 is the root of $x-1$, this means that $\zeta,\zeta^2,\zeta^3,\zeta^4$ must be the roots of $m(x)$; so $m(x)=(x-\zeta)(x-\zeta^2)(x-\zeta^3)(x-\zeta^4)$.

1. By #2, $m(x)$ has no real roots, so it has no roots in $\mathbb{Q}$, so it has no linear factors. If $m(x)$ factors into quadratic factors in $\mathbb{Q}[x]$, then such a factorization is obtainable in $\mathbb{Z}[x]$ and

$$m(x)=x^4+x^3+x^2+x+1=(x^2+ax\pm 1)(x^2+bx\pm 1)$$

where $a+b=1$. Now both the '$\pm$' signs must be positive and

$$\begin{aligned} m(x)=x^4+x^3+x^2+x+1 &= (x^2+ax+1)(x^2+(1-a)x+1) \\ &= x^4+x^3+(2+a-a^2)x^2+x+1. \end{aligned}$$

However, this has only irrational solutions $a=\frac{1\pm\sqrt{5}}{2}$, a contradiction. So $m(x)$ is irreducible in $\mathbb{Q}[x]$.

If you studied Eisenstein's Criterion in a previous Ring Theory course, this method also shows that $m(x)$ is irreducible in $\mathbb{Q}[x]$.

3. The four roots of $m(x)$ are permuted by σ as $\zeta\mapsto\zeta^2\mapsto\zeta^4\mapsto\zeta^3\mapsto\zeta$, so

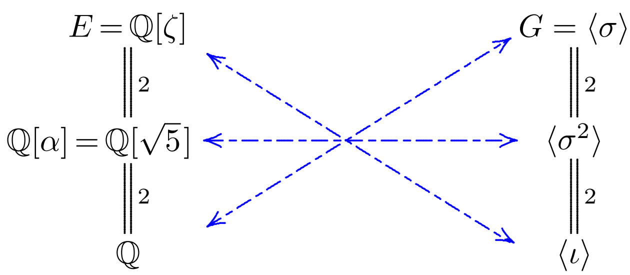
$$\begin{aligned} \sigma(a+b\zeta+c\zeta^2+d\zeta^3) &= a+b\zeta^2+c\zeta^4+d\zeta \\ &= a+b\zeta^2-c(1+\zeta+\zeta^2+\zeta^3)+d\zeta \\ &= (a-c)+(d-c)\zeta+(b-c)\zeta^2-c\zeta^3. \end{aligned}$$

4. From $\alpha^2=(\zeta+\zeta^4)^2=2+\zeta^2+\zeta^3=1-\zeta-\zeta^4=1-\alpha$, we see that α is a root of $g(x)=x^2+x-1$. Thus $\alpha=\frac{1}{2}\big(-1\pm\sqrt{5}\big)$. Since $\alpha\notin\mathbb{Q}$, $g(x)$ is irreducible in $\mathbb{Q}[x]$; so it is the minimal polynomial of α. Using de Moivre's formula, $\alpha=2\cos\frac{2\pi}{5}>0$, so we must have the positive root of $g(x)$, i.e. $\alpha=\frac{1}{2}\big(-1+\sqrt{5}\big)$. This shows that α is real; but this was already evident from the start since $\alpha=\zeta+\overline{\zeta}$, which is twice the real part of ζ.

5. Since $\alpha=\frac{1}{2}\big(-1+\sqrt{5}\big)\in\mathbb{Q}[\sqrt{5}]$, we have $\mathbb{Q}[\alpha]\subseteq\mathbb{Q}[\sqrt{5}]$. But also, $\sqrt{5}=1+2\alpha\in\mathbb{Q}[\alpha]$, so $\mathbb{Q}[\sqrt{5}]\subseteq\mathbb{Q}[\alpha]$. Thus equality holds: $\mathbb{Q}[\alpha]=\mathbb{Q}[\sqrt{5}]$.

Also, $\sigma(\alpha) = \sigma(\zeta+\zeta^4) = \zeta^2+\zeta^3 = -1-\zeta-\zeta^4 = -1-\alpha$. So $\sigma(\sqrt{5}) = \sigma(1+2\alpha) = 1+2(-1-\alpha) = -1-2\alpha = -\sqrt{5}$.

6. The Hasse diagrams of $E = \mathbb{Q}[\zeta]$ and $G = \operatorname{Aut} E = \langle\sigma\rangle = \{\iota, \sigma, \sigma^2, \sigma^2\}$ are



It is easiest to first enumerate the subgroups of G, since G is just a cyclic group of order 4. Next, match each subgroup of G with its fixed field in E. For example, $\sigma^2(\sqrt{5}) = \sigma(\sigma(\sqrt{5})) = \sigma(-\sqrt{5}) = \sqrt{5}$ which generates the intermediate subfield $\mathbb{Q}[\sqrt{5}]$. The Galois correspondence is indicated by the blue arrows.

7. From #4, $\alpha = 2\cos\frac{2\pi}{5} = \frac{1}{2}\big(-1+\sqrt{5}\big)$, so $\cos 72° = \cos\frac{2\pi}{5} = \frac{\alpha}{2} = \frac{1}{4}(-1+\sqrt{5}) \approx 0.30902$.