HW2

Due 5:00pm, Monday, October 21, 2024, on WyoCourses

Instructions: Show your work, and *check* answers whenever possible. Submit solutions through WyoCourses. See the syllabus and FAQ for general expectations regarding homework.

Let $\zeta = e^{2\pi i/5}$. Note that ζ is a root of $x^5-1 = (x-1)m(x)$ where $m(x) = x^4+x^3+x^2+x+1$, and $\zeta \neq 1$; so ζ is a root of $m(x)$. If you have trouble writing ζ by hand, you may substitute z instead.

1. (*10 points*) Show that $m(x) \in \mathbb{Q}[x]$ is irreducible (so it is the minimal polynomial of ζ over $\mathbb{Q}$).

2. (*10 points*) Show that $m(x) = (x-\zeta)(x-\zeta^2)(x-\zeta^3)(x-\zeta^4)$.

Now consider the field extension $E = \mathbb{Q}[\zeta] \supset \mathbb{Q}$ of degree $[E:\mathbb{Q}] = 4$. Since E contains all the roots of $m(x)$, it is a normal extension. In particular, E has 4 automorphisms. The group $G = \operatorname{Aut} E$ is cyclic of order 4; indeed, $G = \langle\sigma\rangle = \{\iota, \sigma, \sigma^2, \sigma^3\}$ where $\sigma(\zeta) = \zeta^2$.

3. (*10 points*) How does σ permute the four roots of $m(x)$? With respect to the basis $\{1, \zeta, \zeta^2, \zeta^3\}$ of $E \supset \mathbb{Q}$, evaluate

$$\sigma(a + b\zeta + c\zeta^2 + d\zeta^3) = \boxed{} + \boxed{}\,\zeta + \boxed{}\,\zeta^2 + \boxed{}\,\zeta^3$$

   where the blanks are rational numbers involving $a, b, c, d \in \mathbb{Q}$.

4. (*20 points*) Find the minimum polynomial of $\alpha = \zeta + \zeta^4$ over $\mathbb{Q}$. Is α real? Explain.

5. (*20 points*) Using #4, show that $\sqrt{5} \in E$ and $\sigma(\sqrt{5}) = -\sqrt{5}$.

6. (*20 points*) Find all subfields of E, and all subgroups of G. Sketch their Hasse diagrams and illustrate the one-to-one correspondence between subfields of E and subgroups of G.

7. (*10 points*) Using the steps above (and de Moivre's formula), show that $\cos 72° = \frac{1}{4}(-1+\sqrt{5})$.