University of Wyoming — Department of Mathematics

Math 4520—Fall 2024

# Algebra III: Fields

$F[\alpha] \cong F[t]/(f(t))$

## Solutions to HW1

1. If $f(x)$ is reducible in $\mathbb{Q}[x]$, then $f(x)$ is reducible in $\mathbb{Z}[x]$ and so factors as $f(x) = (x-a)(x^2+bx+c)$ for some $a,b,c \in \mathbb{Z}$; but then $ac = 3$ and $f(x)$ has a root $a \in \{-3,-1,1,3\}$. However, $f(a) = -24,-2,-4,18$ for $a = -3,-1,1,3$ respectively. This is a contradiction; so in fact $f(x) \in \mathbb{Q}[x]$ must be irreducible.

2. The polynomial $f(x)$ has exactly one real root, using methods from Calculus I. From the sign of $f'(x) = 3x^2 - 2 = 3\big(x+\sqrt{2/3}\,\big)\big(x-\sqrt{2/3}\,\big)$, we see that $f$ is increasing on $\big(-\infty,-\sqrt{2/3}\,\big)$ and on $\big(\sqrt{2/3}\,,\infty\big)$, and decreasing on $\big(-\sqrt{2/3}\,,\sqrt{2/3}\,\big)$. Since $f(-\sqrt{2/3}\,) = \frac{4\sqrt{6}-27}{6} < 0$, $f$ is negative on $(-\infty,\sqrt{2/3}\,]$. Since $f$ is increasing on $(\sqrt{2/3}\,,\infty)$, it has at most one root on that interval. However, we have seen (in #1) that $f$ changes sign, so it has exactly one real root.

3. From $\theta^3 = 2\theta+3$ we obtain $\theta^4 = 2\theta^2+3\theta$.
   (a) $\alpha+\beta = 2\theta^2+\theta-2$.
   (b) $\alpha-\beta = -\theta-4$.
   (c) $\alpha\beta = \theta^4+\theta^3-2\theta^2-3\theta-3 = (2\theta^2{+}3\theta)+(2\theta{+}3)-2\theta^2-3\theta-3 = 2\theta$.
   (d) $\frac{\alpha}{\beta} = a\theta^2{+}b\theta{+}c$ yields
   $$\begin{aligned}\alpha &= (a\theta^2{+}b\theta{+}c)(\theta^2{+}\theta+1)\\ &= (3a{+}b{+}c)\theta^2+(5a{+}3b{+}c)\theta+(3a{+}3b{+}c).\end{aligned}$$
   Solving this system of three linear equations gives the unique solution $(a,b,c) = \big(\frac{3}{2},\,-2,\,-\frac{3}{2}\big)$; thus $\frac{\alpha}{\beta} = \frac{3}{2}\theta^2-2\theta-\frac{3}{2}$.

4. We have $\alpha^2 = (\theta^2{-}3)^2 = (2\theta^2{+}3\theta){-}6\theta^2{+}9 = -4\theta^2{+}3\theta{+}9$; and similarly, $\alpha^3 = (-4\theta^2{+}3\theta{+}9)(\theta^2{-}3) = 13\theta^2{-}15\theta{-}18$. We seek $a,b,c \in \mathbb{Q}$ such that
   $$\alpha^3+a\alpha^2+b\alpha+c = (13\theta^2{-}15\theta{-}18)+a(-4\theta^2{+}3\theta{+}9)+b(\theta^2{-}3)+c = 0.$$
   Collecting terms, we obtain a linear system in $a,b,c$, which has a unique solution $(a,b,c) = (5,7,-6)$. Thus the minimal polynomial of $\alpha$ over $\mathbb{Q}$ is $x^3+5x^2+7x-6$.

*Check*: We may evaluate the expressions in #3,4 using any of the three roots of $f(x)$. For convenience, we try the real root $\theta \approx 1.893289196$, $\alpha \approx 0.58454398$, $\beta \approx 6.477833176$. This gives excellent numerical agreement for our answers in #3,4.

5. By the method described in class, we obtain a companion matrix for $f(x)$, namely $A = \begin{bmatrix}0&0&3\\1&0&2\\0&1&0\end{bmatrix}$. By construction, the characteristic polynomial of $A$ is $\det(xI-A) = f(x)$; and so by the Cayley-Hamilton Theorem, $f(A) - 0$. An explicit isomorphism $F \to K$, $\mathbb{Q}[\theta] \to \mathbb{Q}[A]$ is determined by $\theta \mapsto A$. This is the map $a+b\theta+c\theta^2 \mapsto aI+bA+cA^2$ for all $a,b,c \in \mathbb{Q}$.