University of Wyoming

Math 4520—Fall 2024

# Algebra III

Fields

Department of Mathematics

## HW1

(Due 5:00pm Friday, September 27, 2024)

> *Instructions:* Work by hand and calculator showing your work, and checking your answers whenever reasonably possible. If you have access to computer software, this may be used to *check* answers. Submit solutions through WyoCourses. See the syllabus and FAQ for general expectations regarding homework.

In the following, $f(x) = x^3 - 2x - 3 \in \mathbb{Q}[x]$.

1. (*10 points*) Show that $f(x)$ is irreducible in $\mathbb{Z}[x]$ (and hence also in $\mathbb{Q}[x]$).

2. (*10 points*) How many real roots does $f(x)$ have? Explain.

Now let $\theta \in \mathbb{C}$ be any root of $f(x)$. (You are given this assumption, but you are not told which of the roots of $f(x)$ it is, nor is that information relevant.) From the general theory, we know that

$$F = \mathbb{Q}[\theta] = \{a + b\theta + c\theta^2 \,:\, a, b, c \in \mathbb{Q}\}$$

is a field; and that every element of $F$ is uniquely representable in the standard form $a + b\theta + c\theta^2$ with $a, b, c \in \mathbb{Q}$. (We are saying that $F \supset \mathbb{Q}$ is a cubic extension, i.e. an extension field of degree $[F : \mathbb{Q}] = 3$. For now, you may take these facts as given; in due time, you should be able to justify them.)

Fix the elements $\alpha, \beta \in F$ given by $\alpha = \theta^2 - 3$, $\beta = \theta^2 + \theta + 1$.

3. (*20 points*) Compute each of the following in $F$, simplifying your answers and expressing them in the standard form for elements of $F$:
   (a) $\alpha + \beta$
   (b) $\alpha - \beta$
   (c) $\alpha\beta$
   (d) $\alpha/\beta$

4. (*20 points*) Find the unique monic polynomial $g(x) \in \mathbb{Q}[x]$ of smallest degree having $\alpha$ as a root.

A reasonable way to check your answers in #3,4 is to find a real root of $f(x)$, correct to as many decimal places as your calculator can supply, and then evaluate your expressions in #3,4 using these decimal approximations. Because of the inaccuracy of your floating point approximations, you should find that your answers do not agree perfectly with the true values expected; but the agreement should be so close that this will provide a very good check. It cannot matter which root of $f(x)$ you choose for this purpose, as we will discuss later.

5. (*20 points*) Let $\mathbb{Q}^{3\times 3}$ be the ring of all $3 \times 3$ matrices with entries in $\mathbb{Q}$. Find a subring $K \subset \mathbb{Q}^{3\times 3}$ isomorphic to $F = \mathbb{Q}[\theta]$. You should in fact find an explicit matrix $A \in \mathbb{Q}^{3\times 3}$ such that $K = \mathbb{Q}[A]$, and an explicit isomorphism $F \to K$ mapping $\theta \mapsto A$.