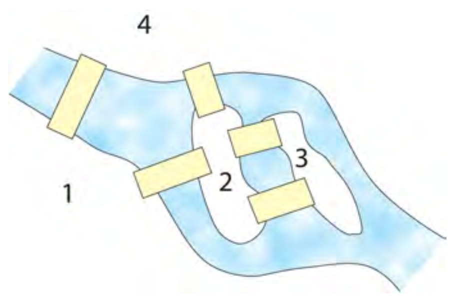
Counting Walks

We count the number of walks of length n between any two vertices in a graph.

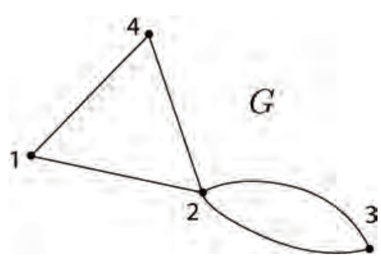
Consider the following map with four land masses (labeled by the elements of the set $[4] := \{1, 2, 3, 4\}$) and five bridges (not labeled). For each $i, j \in [4]$, denote by $w_n(i, j)$ the number of walks of length from land mass i to land mass j, where the *length* of a walk is the number of bridges crossed during the walk. A table of values of $w_2(i, j)$, the number of walks of length 2 from vertex i to vertex j, is shown:



$w_2(i, j)$

$i \backslash j$	1	2	3	4
1	2	1	2	1
2	1	6	0	1
3	2	0	4	2
4	1	1	2	2

We represent this map using the graph G whose vertices represent land masses, and whose edges represent bridges:



so that $w_n(i, j)$ is the number of walks of length n from vertex i to vertex j in G. The number of walks of length n in a graph G is simply expressed using matrix arithmetic, as we know explain.

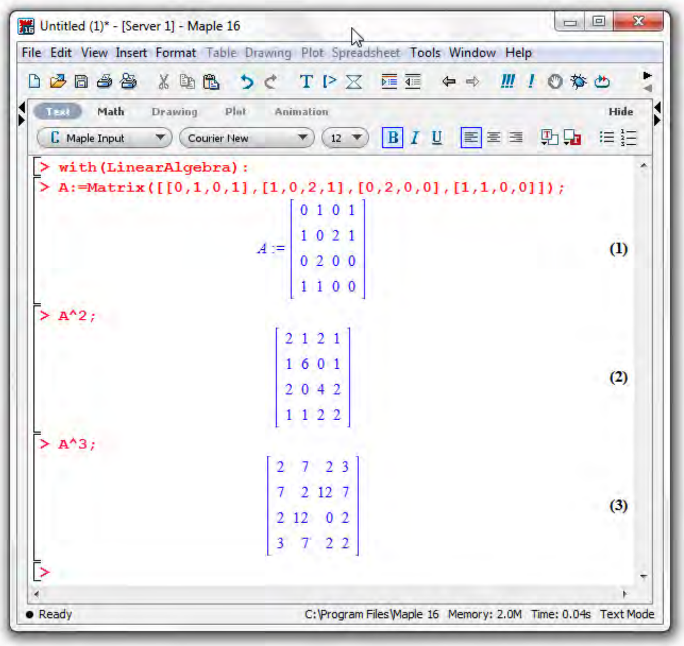
Let G be a graph on m vertices. We may assume that the vertices are indexed using the elements of $[m] = \{1, 2, 3, \ldots, m\}$. The *adjacency matrix* of G is the $m \times m$ matrix whose (i, j)-entry is

$$a_{ij} = \text{number of edges from vertex } i \text{ to vertex } j.$$

In a simple graph, where no loops or multiple edges are allowed, we have $a_{ij} = 0$ or 1, and $a_{ii} = 0$; but for our present purposes, such a restriction is not needed. A *walk of length n* *in* G from vertex v to vertex w, is a sequence of n edges, starting with an edge from vertex $v_0 = v$ to some vertex v_1, followed by an edge from v_1 to a vertex v_2, etc., and ending with an edge from vertex v_{n-1} to vertex $v_n = w$. In a walk, repeated vertices and repeated edges are permitted (compare with a *trail*, where vertices may be repeated but not edges).

> **Theorem 1.** The number $w_n(i, j)$ of walks of length n from vertex i to vertex j in a graph G, is the (i, j)-entry of A^n where A is the adjacency matrix of G.

The following Maple® code demonstrates using our graph G above, where we first enter the adjacency matrix A and then computes A^2 and A^3:



Note that the matrix A^2 yields our table of values of $w_2(i, j)$ above; and the number of walks of length 3 from vertex 1 to vertex 4, say, is $w_3(2, 4) = 7$, the $(2, 4)$-entry of A^3.

To see why this works, consider first the case $n = 2$. The number of walks of length 2 from vertex i to vertex j is

$$\begin{aligned} w_2(i,j) &= \sum_{k \in [m]} \binom{\text{number of edges}}{\text{from } i \text{ to } k} \binom{\text{number of edges}}{\text{from } k \text{ to } j} \\ &= \sum_{k \in [m]} a_{ik} a_{kj} \\ &= \text{the } (i,j)\text{-entry of } A^2 \end{aligned}$$

by the definition of matrix multiplication. Similarly for arbitrary n, the number of walks of length n from vertex i to vertex j is

$$\begin{aligned}
w_n(i,j) &= \sum_{v_1,v_2,\ldots,v_{n-1}\in[m]} \binom{\text{number of edges}}{\text{from } i \text{ to } v_1} \binom{\text{number of edges}}{\text{from } v_1 \text{ to } v_2} \cdots \binom{\text{number of edges}}{\text{from } v_{n-1} \text{ to } j} \\
&= \sum_{v_1,v_2,\in[m]} a_{iv_1} a_{v_1v_2} a_{v_2v_3} \cdots a_{v_{n-1}j} \\
&= \text{the } (i,j)\text{-entry of } A^n
\end{aligned}$$

which proves Theorem 1. □

For each pair of vertices (i, j), we can compute as many terms as desired of the sequence $w_0(i, j)$, $w_1(i, j)$, $w_2(i, j)$, $w_3(i, j)$, ... by 'simply' taking successive powers of the adjacency matrix A, then reading off the (i, j) entry. Better yet, we can explicitly obtain the (ordinary) generating function for this sequence,

$$W(x) = W_{ij}(x) = \sum_{n\geq 0} w_n(i,j)x^n = w_0(i,j) + w_1(i,j)x + w_2(i,j)x^2 + w_3(i,j)x^3 + \cdots,$$

sometimes known as the *walk generating function*.

Theorem 2. The generating function $W_{ij}(x)$ for $w_n(i, j)$ equals the (i, j)-entry of $(I - xA)^{-1}$.

Proof. By direct expansion we see that

$$\begin{aligned}
&(I - xA)(I + xA + x^2A^2 + x^3A^3 + x^4A^4 + \cdots) \\
&\qquad = I - xA + xA - x^2A^2 + x^2A^2 - x^3A^3 - x^4A^4 + x^4A^4 - \cdots \\
&\qquad = I,
\end{aligned}$$

so that

$$(I - xA)^{-1} = I + xA + x^2A^2 + x^3A^3 + x^4A^4 + \cdots.$$

The (i, j)-entry of this matrix is $\sum_{n\geq 0} x^n w_n(i, j) = W_{ij}(x)$. □

In our original example, the generating function for the number of walks of length n from vertex i to vertex j is the (i, j)-entry of

$$(I - xA)^{-1} = \frac{1}{d(x)} \begin{bmatrix} 1-5x^2 & x(1+x) & 2x^2(1+x) & x(1+x-4x^2) \\ x(1+x) & 1-x^2 & 2x(1-x^2) & x(1+x) \\ 2x^2(1+x) & 2x(1-x^2) & (1+x^2)(1-2x) & 2x^2(1+x) \\ x(1+x-4x^2) & x(1+x) & 2x^2(1+x) & 1-5x^2 \end{bmatrix}$$

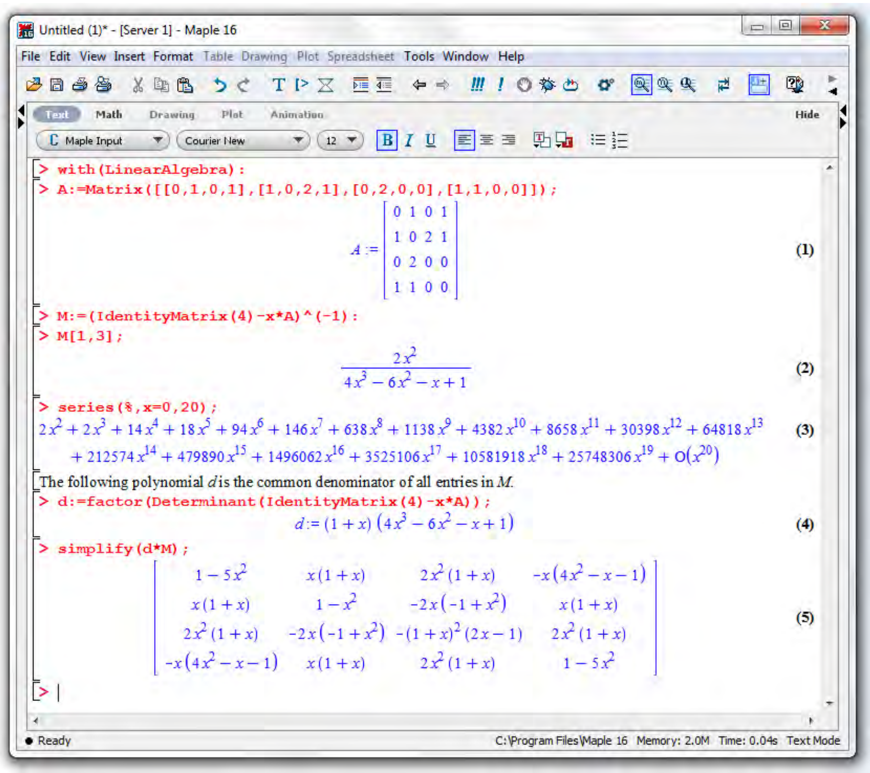
where the common denominator $d(x) = (1 + x)(1 - x - 6x^2 + 4x^3)$. In particular

$$W_{13}(x) = 2x^2 + 2x^3 + 14x^4 + 18x^5 + 94x^6 + 146x^7 + 638x^8 + 1138x^9 + 4382x^{10} + \cdots,$$

so the number of walks of length n from vertex 1 to vertex 3 is given by

$$0, 0, 2, 2, 14, 18, 94, 146, 638, 1138, 4382, \ldots$$

for $n = 0, 1, 2, 3, \ldots$. All these computations are demonstrated in the Maple® session



This concludes the solution of the 4-part problem Will solved in the 1997 film *Good Will Hunting*.