University of Wyoming — Math 3700—Spring 2023 — Combinatorics — Department of Mathematics

Optional Test 2

April, 2023

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an $8.5'' \times 11''$ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Time permitted: 60 minutes. There are 15 bonus points available, but the maximum possible score is 100 points.

1. Assuming M&M's are manufactured in five different colors (red, blue, yellow, green and orange).

   (a) *(10 points)* How many ways can we select a handful of nine M&M's from this supply?

   (b) *(10 points)* How many ways can we select a handful of nine M&M's such that at least one M&M of each color appears in the selected handful?

2. Let a_n be the number of 01-free bitstrings of length n, i.e. the number of strings $b_1 b_2 b_3 \cdots b_n$ with every $b_i \in \{0, 1\}$, in which there is no '0' followed *immediately* by '1' (there is no $i \in \{2, 3, \ldots, n\}$ satisfying $b_{i-1}{=}0$ and $b_i{=}1$).

   (a) *(5 points)* Tabulate a_n for $n \in \{0, 1, 2, 3, 4, 5\}$.

   (b) *(10 points)* Express the generating function $\sum_{n=0}^{\infty} a_n x^n$ in simple closed form as a rational function of x.

   (c) *(10 points)* Give an example of a graph Γ of order 2 with vertices '1' and '2', such that a_n may be interpreted as the number of walks of length n in Γ starting at vertex 1. (The graph Γ is also called a *finite state machine*. And like the example given in class for 11-free bitstrings, this graph Γ will not be an *ordinary* graph: here we allow loops, directed edges, and possibly multiple edges.)

   Hint: It will help to think of vertex 1 as the state in which our partial string (up to this point in the walk) does *not* have last bit '0'; and vertex 2 is the state in which it *does* have last bit '0'.)

3. A simple betting game requires the player to toss a fair coin $2n$ times. Each toss results in an outcome of H or T (heads or tails), each with probability 50%. The player starts the game with no money; and every time the coin comes up heads, the player wins \$1. Every time the toss comes up tails, the player loses \$1.

   For example, George tries this game and lasts 10 rounds with the outcome HHTHTHTTTH, at which point he stops after breaking even. At one point (on the ninth round) he went \$1 into debt after getting tails, but he recovered on the tenth round with his final toss of heads, at which point he decided to stop. His net gain was zero (after winning \$5 and losing \$5, all of which cancelled).

   (a) *(5 points)* How many ways is it possible for a player to continue for $2n$ rounds? That is, how many sequences of H and T of length $2n$ are possible?

   (b) *(5 points)* If a player continues for $2n$ rounds, what fraction represents the probability that he will break even (and walk away without any net gain or loss, like George did)?

   (c) *(5 points)* If a player continues for $2n$ rounds, what is the probability that he will make money (and walk away with a positive net gain)?

   (d) *(5 points)* If a player continues for $2n$ rounds, what is the probability he will break even without ever going into debt? (This means that he had n heads and n tails, but at no time during the game did the number of tails ever exceed the number of heads).

4. *(20 points)* Simplify each of the following sums. If you prefer, you may express your answers in a symbolic way (e.g. $2^{10}\binom{10}{5}$ rather than 258,048).

(a) $\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \cdots + \binom{10}{9} + \binom{10}{10}$

(b) $\binom{10}{0} - \binom{10}{1} + \binom{10}{2} - \binom{10}{3} + \cdots - \binom{10}{9} + \binom{10}{10}$

(c) $\binom{10}{0} + 2\binom{10}{1} + 2^2\binom{10}{2} + 2^3\binom{10}{3} + \cdots + 2^9\binom{10}{9} + 2^{10}\binom{10}{10}$

(d) $0\binom{10}{0} + 1\binom{10}{1} + 2\binom{10}{2} + 3\binom{10}{3} + \cdots + 9\binom{10}{9} + 10\binom{10}{10}$

5. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) Every function $[n] \to [k]$ is either a surjection or an injection (but not both, unless $n = k$). ______*(True/False)*

(b) If the power series $\sum_{n=0}^{\infty} a_n x^n$ is a rational function, then the sequence of coefficients a_n satisfies a linear recurrence. ______*(True/False)*

(c) Given a vertex x of degree k in an ordinary graph Γ, there are exactly k walks of length 2 in Γ starting and ending at vertex x . ______*(True/False)*

(d) The number of ternary strings of length n (i.e. strings of length n over the ternary alphabet $\{0, 1, 2\}$) has generating function $\frac{1}{1-3x}$. ______*(True/False)*

(e) The sequence of binomial coefficients $\binom{n+20}{n}$ (for $n = 0, 1, 2, 3, \ldots$) grows at an exponential rate. ______*(True/False)*

(f) If m is any positive integer, then the binomial expansion of $(1 - x)^{-m}$ is a power series, all of whose coefficients are non-negative integers. ______*(True/False)*

(g) If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$, then the generating function for the sequence a_0+b_0, a_1+b_1, a_2+b_2, $\ldots$ is $A(x)+B(x)$. ______*(True/False)*

(h) The total number of walks of length n in K_n is a polynomial in n of degree n. (Recall that K_n denotes the complete graph of order n.) ______*(True/False)*

(i) If $F(x)$ is the generating function for the number of 11-free bitstrings of length n, then $F(x)$ grows to infinity at an exponential rate as $x \to \infty$. ______*(True/False)*

(j) Let A be the adjacency matrix of an undirected graph Γ of order n. Then $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A. ______*(True/False)*