University of Wyoming

Math 3700—Spring 2023

Combinatorics

Department of Mathematics

Solutions to Optional Test 2

April 2023

1. (a) $\binom{9+5-1}{5-1} = \binom{13}{4} = 715$

   (b) First take one M&M of each color, then select four more M&M's at random. There are $\binom{4+5-1}{5-1} = \binom{8}{4} = 70$ ways to do this.

2. (a) $a_n = 1, 2, 3, 4, 5, 6$ for $n = 0, 1, 2, 3, 4, 5$ respectively. For example, the four 01-free bitstrings of length three are 000, 100, 110, 111. In general, $a_n = n+1$.

   (b) $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1)x^n = \frac{d}{dx} \sum_{n=0}^{\infty} x^{n+1} = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}$. Alternatively, $1 + 2x + 3x^2 + 4x^3 + \cdots = (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots) = \frac{1}{(1-x)^2}$.

   (c) The graph 

has the required properties. Its adjacency matrix is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has exactly $n+1$ walks starting at vertex 1. As each walk is traversed, a 01-free bitstring of length n is printed out; and this output uniquely identifies the walk that it came from. From $W(x) = (I - xA)^{-1} = \begin{bmatrix} 1-x & -x \\ 0 & 1-x \end{bmatrix}^{-1} = \frac{1}{(1-x)^2}\begin{bmatrix} 1-x & x \\ 0 & 1-x \end{bmatrix}$, we add the two entries in the first row to get $w(x) = \frac{1}{(1-x)^2}$; and this gives yet another proof of the formula for the generating function.

3. (a) 2^{2n}

   (b) There are $\binom{2n}{n}$ strings of length $2n$ over the binary alphabet {H,T} having n of each letter. The probability that the player breaks even is $\frac{1}{2^{2n}}\binom{2n}{n}$.

   (c) The probability of *not* breaking even is $1 - \frac{1}{2^{2n}}\binom{2n}{n}$. By symmetry, this equals twice the probability of making money. So the probability of making money is $\frac{1}{2}\left[1 - \frac{1}{2^{2n}}\binom{2n}{n}\right]$ (which is the same as the probability of losing money).

   (d) The sequences of H's and T's for which the player never goes into debt and finishes after $2n$ coin flips by breaking even, correspond exactly to Dyck paths of length $2n$ if one replaces H and T by E and N respectively. The number of such sequences is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. The probability of obtaining such a sequence by chance with $2n$ coin flips, is $\frac{C_n}{2^{2n}} = \frac{1}{2^{2n}(n+1)}\binom{2n}{n}$.

4. Use $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

   (a) $(1+1)^{10} = 2^{10} = 1024$

   (b) $(1-1)^{10} = 0^{10} = 0$

   (c) $(1+2)^{10} = 3^{10} = 59{,}049$

(d) Let $S = 0\binom{10}{0} + 1\binom{10}{1} + 2\binom{10}{2} + 3\binom{10}{3} + \cdots + 10\binom{10}{10}$
$= 10\binom{10}{10} + 9\binom{10}{9} + 8\binom{10}{8} + 7\binom{10}{7} + \cdots + 0\binom{10}{0}$.

Adding these two expressions for S and using the identity $\binom{10}{k} = \binom{10}{10-k}$ gives

$$2S = 10\left[\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \cdots + \binom{10}{10}\right] = 10\cdot 2^{10} = 10240$$

and so $S = 5{,}120$.

5. (a) F (b) T (c) T (d) T (e) F (f) T (g) T (h) T (i) F (j) T

Here are some remarks and partial explanations for answers in #5:

(a) The constant function $f : [2] \to [2]$ defined by $f(1) = f(2) = 1$ is neither injective nor surjective.

(b) Suppose $\sum_{k=0}^{\infty} a_k x^k = \frac{g(x)}{1-c_1x-c_2x^2-\cdots-c_dx^d}$. We cross-multiply to clear the denominator, giving $(1 - c_1x - c_2x^2 - \cdots - c_dx^d)(a_0 + a_1x + a_2x^2 + \cdots) = g(x)$. For all $n \geqslant \max\{d, 1+\deg g(x)\}$, comparing coefficients of x^n on both sides gives $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_da_{n-d}$.

(c) The walks of length 2 from x to x all have the form $x \sim y \sim x$ where y ranges over the k neighbors of x.

(d) The number of ternary strings of length n is 3^n, and the generating function for this sequence is the geometric series $\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n$.

(e) The binomial coefficient $\binom{n+20}{n} = \binom{n+20}{20} = \frac{1}{20!}(n+1)(n+2)\cdots(n+20)$ is a polynomial in n (of degree 20).

(f) Recall that $(1 - x)^{-m} = \sum_{k=0}^{\infty} \binom{-m}{k}(-x)^k = \sum_{k=0}^{\infty} \binom{m+k-1}{k}x^k$, as shown in class. (This is the generating function for one of our important sequences.)

(g) $\sum_{n=0}^{\infty} (a_n+b_n)x^n = \sum_{n=0}^{\infty} a_nx^n + \sum_{n=0}^{\infty} b_nx^n$.

(h) There are n choices for the first vertex in a walk, and $n-1$ choices for each of the remaining vertices, so altogether $n(n-1)^{n-1}$ walks of length n in K_n.

(i) The generating function $F(x) = \frac{1+x}{1-x-x^2} \to 0$ as $x \to \infty$. Even if you don't remember the exact form of the generating function in this case, there is *no* rational function $F(x)$ which grows at an exponential rate as $x \to \infty$.

(j) This is one of the main assertions of the Spectral Theorem.