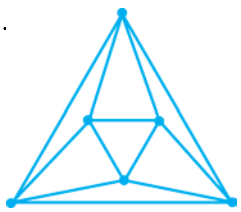
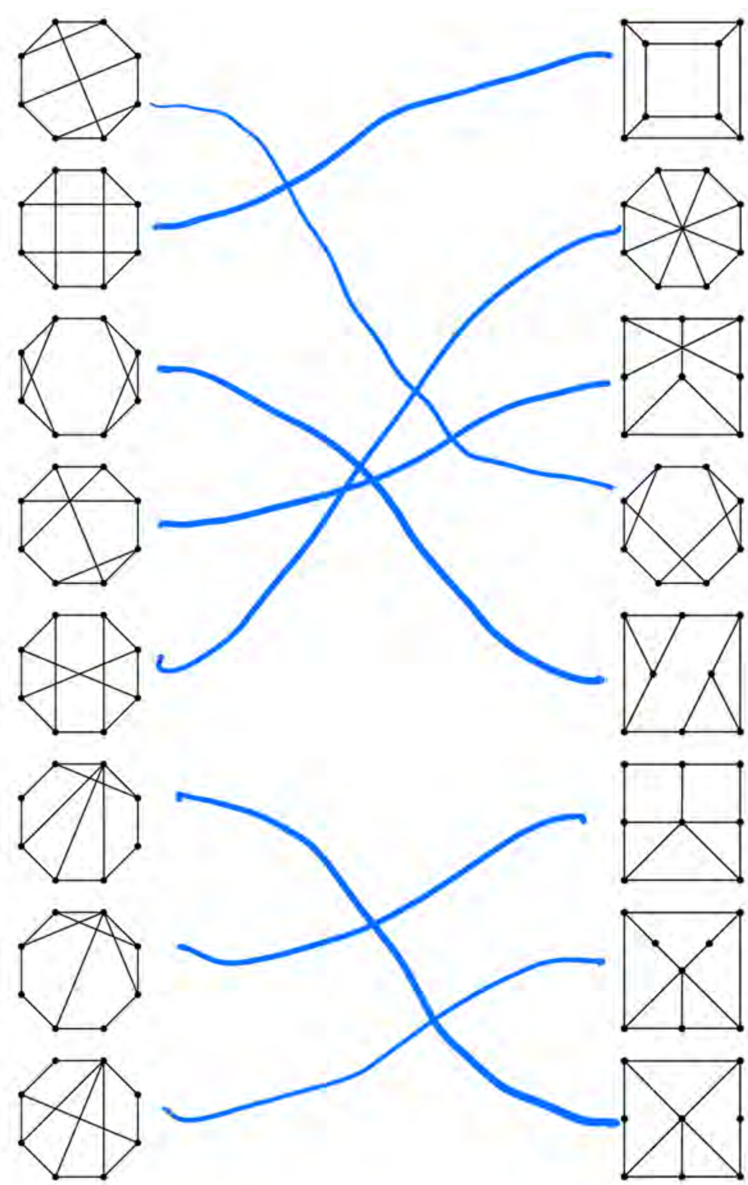
Solutions to Test 1

March 2023

1.

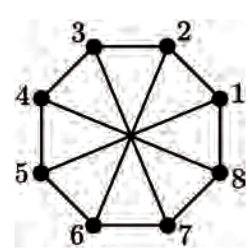


2.



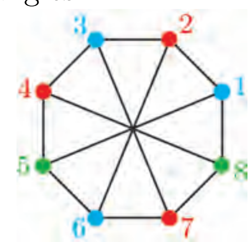
3. (a) The diameter is 2. For example, vertices 1 and 3 are at distance 2.

   (b) The girth is 4. An example of a shortest circuit (cycle) is $(1, 2, 6, 5, 1)$.



(c) The clique number is $\omega(\Gamma) = 2$. The graph has edges but no triangles.

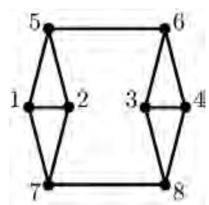
(d) The coclique number is $\alpha(\Gamma) = 3$. For example, $\{1, 3, 6\}$ is a coclique of size 3.



(e) The chromatic number is $\chi(\Gamma) = 3$. For example, we may properly color 1,3,6 blue; 2,4,7 red; and 5,8 green as shown.

(f) The number of automorphisms is $|\text{Aut}\,\Gamma| = 16$. These automorphisms are the same group as the symmetry group of a regular octagon, which is a dihedral group of order 16, containing eight rotational and eight reflective symmetries.

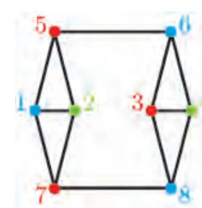
4. (a) Yes, Γ is planar:



(b) Γ has 16 automorphisms. We have $\text{Aut}\,\Gamma = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle = \{\sigma_1^{i_1}\sigma_2^{i_2}\sigma_3^{i_3}\sigma_4^{i_4} : i_1, i_2, i_3, i_4 \in \{0, 1\}\}$ where $\sigma_1 = (57)(68)$ is the reflection across the horizontal axis of symmetry; $\sigma_2 = (14)(23)(56)(78)$ is the reflection across the vertical axis of symmetry; $\sigma_3 = (12)$; and $\sigma_4 = (34)$.

(c) The diameter of Γ is 3. For example, vertices 1 and 3 are at distance 3, the maximum possible.

(d) The chromatic number is $\chi(\Gamma) = 3$. On the one hand, $\chi(\Gamma) \geqslant 3$ because Γ contains triangles. And on the other hand, it is easy to properly 3-color the vertices of Γ as shown.



5. (a) T (b) F (c) F (d) T (e) F (f) F (g) F (h) F (i) F (j) T

Here are some remarks and partial explanations for answers in #5:

(a) The Hamilton circuit in H_n we described in class for $n = 2, 3$ (using a Gray code) works for all $n \geqslant 2$. This can be shown by induction.

(b) K_n is regular of odd degree for $n = 2, 4, 6, 8, \ldots$. A graph cannot have an Euler circuit unless all its vertices have even degree.

(c) If there are 7 vertices, then there are $\binom{7}{2} = 21$ pairs of vertices. If Γ is a graph of order 7 with e edges, then its complement $\overline{\Gamma}$ has $21-e$ edges. Of course, $e \neq 21-e$.

(d) The easiest example (and the one that was described in class) is the 'random graph' R, also known as the Erdős-Rényi graph or Rado graph. Recall that this is the graph constructed on an infinite vertex set, where we flip a coin for each vertex pair, to determine whether or not we will put an edge. This 'always' (i.e. 100% of the time) gives the same graph R, and its complement satisfies $\overline{R} \cong R$.

(e) There are many counterexamples. We have mentioned the 5-cycle, whose complement is isomorphic to itself; here both the graph and its complement are connected.

(f) Here the sum of the terms in the sequence is odd, whereas the sum of the vertex degrees for any finite graph must be even.

(g) Such a graph would have order 8; and with the vertex of order 7, we see that the graph would be connected. This contradicts having an isolated vertex (a vertex of degree zero). This argument was discussed in class.

(h) The Petersen Graph has 120 automorphisms, which is a lot, but there are still many graphs of order 10 with more than that. For example, the complete graph K_{10} has $10! = 3{,}628{,}800$ automorphisms.

(i) Recall that $R(3,3) = 6$: for any graph Γ of order $n \geqslant 6$, either Γ or $\overline{\Gamma}$ contains a triangle.

(j) Every graph with at most 8 edges is planar, since a nonplanar graph must involve either K_5 (which has 10 edges) or $K_{3,3}$ (which has 9 edges). Moreover, every connected graph of order n with only $n-1$ edges is 'just barely' connected; it is a 'tree' and hence planar, as follows by induction. Some examples are