Solutions to Sample Test 2

April, 2023

1. Here $a_n = \binom{2n}{n}$.

   (a) $\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$ as shown in class and on the handout on Catalan numbers.

   (b) $a_0 = 1$ is divisible by 1;
   $a_1 = 2$ is divisible by 2;
   $a_2 = 6$ is divisible by 3;
   $a_3 = 20$ is divisible by 4;
   $a_4 = 70$ is divisible by 5;
   $a_5 = 252$ is divisible by 6.

   (c) Yes, a_n is divisible by $n+1$ for all $n \geqslant 0$. This is known because the Catalan numbers C_n are integers satisfying $(n+1)C_n = a_n$.

2. (a) $5! = 120$

   (b) *First Solution.* We denote the image of $f : A \to B$ by $f(A) = \{f(a) : a \in A\} \subseteq B$. There are just 3 choices of constant function $f([5]) = \{r\}$ with $r \in [3]$. For functions having exactly two values $f([5]) = \{r, s\}$ with $r \neq s$ in $[3]$, there are $\binom{3}{2} = 3$ ways to choose the pair $\{r, s\}$, then $2^5 - 2 = 30$ ways to construct the function f onto $\{r, s\}$ (here we don't consider the 2 constant functions with value r or s, which we have already counted). This gives $3 \cdot 30 = 90$ choices of function f whose image has size $|f([5])| = 2$. Altogether there are $3^5 = 243$ functions $f : [5] \to [3]$, so this leaves $243 - 3 - 90 = 150$ surjections.

   Second Solution. The table of values of a surjective function $f : [5] \to [3]$ has one of the two forms

x	a	b	c	d	e
$f(x)$	r	r	r	s	t

or

x	a	b	c	d	e
$f(x)$	r	r	s	s	t

   where $[5] = \{1, 2, 3, 4, 5\} = \{a, b, c, d, e\}$ and $[3] = \{r, s, t\}$ (permuting the five elements of the domain in some order, and similarly the three elements of the range). In the first case there are $\binom{5}{3} = 10$ ways to choose the 3-subset $\{a, b, c\} \subset [5]$, then 3 ways to choose the function value $r \in [3]$ on this subset, and 2 ways to choose which way to match the remaining elements d, e of the domain, to the remaining two elements s, t of the range. This is $10 \cdot 3 \cdot 2 = 60$ choices for a surjection f of the first type.

In the second case, there are 5 ways to choose $e \in [5]$ and 3 ways to choose $t = f(e)$. The remaining four elements of [5] can be paired up as $\{a, b\}$, $\{c, d\}$ in 3 different ways, and then the remaining two elements $r, s \in [3]$ can be assigned to these pairs in 2 different ways. This makes $5 \cdot 3 \cdot 3 \cdot 2 = 90$ surjections having a table of values of the second type. This leaves $60 + 90 = 150$ surjections $[5] \to [3]$.

In class I mentioned that it is trivial to count functions $[n] \to [k]$, and just a little bit harder to count injections $[n] \to [k]$, but it is surprisingly tricky to count surjections $[n] \to [k]$. Problem 2(b) is evidence of that. The cleanest formula for counting surjections uses Stirling numbers, which we haven't covered yet, and may not unless we have time. It is only because the values $n = 5$ and $k = 3$ are reasonably small that it makes sense to do this example at all by hand.

(c) $(5)_3 = 5 \cdot 4 \cdot 3 = 60$

(d) $3^5 = 243$

3. $f(x, y, z, w) = (x + 2y + 3z + 4w)^{10} = \sum_{i,j,k,\ell} \binom{10}{i,\, j,\, k,\, \ell} x^i (2y)^j (3z)^k (4w)^\ell$ where the indices i, j, k, ℓ range over all non-negativie integers satisfying $i+j+k+\ell = 10$.

(a) Here we count 4-tuples (i, j, k, ℓ) of non-negative integers adding up to 10. This is the same as counting the number of ways to select 10 M&M's that come in 4 colors, or the number of ways to give 10 identical coins to 4 students; and this number is $\binom{10+4-1}{4} = 715$.

(b) $\binom{10}{3,\, 1,\, 2,\, 4} x^3 (2y)^1 (3z)^2 (4w)^4 = 12600 \cdot 2 \cdot 9 \cdot 256 x^3 y z^2 w^4 = 58{,}060{,}800 x^3 y z^2 w^4$. The coefficient is 58,060,800.

(c) $f(1, 1, 1, 1) = 10^{10} = 10{,}000{,}000{,}000$

4. (a) $\binom{11+8-1}{11} = \binom{18}{7} = 31{,}824$

(b) First give one coin to each student, then distribute the remaining 3 coins. The number of ways to do this is $\binom{3+8-1}{3} = \binom{10}{3} = 120$.

(c) $8^{11} = 8{,}589{,}934{,}592$

5. (a) The sequence $1, 1, 2, 4, 8, 16, 32, \ldots$ is defined by $a_n = \begin{cases} 1, & \text{if } n = 0; \\ 2^{n-1}, & \text{if } n \geqslant 1. \end{cases}$

(b) $1 + x + 2x^2 + 4x^3 + 8x^4 + \cdots = 1 + x(1 + 2x + 4x^2 + 8x^3 + \cdots) = 1 + \frac{x}{1-2x} = \frac{1-x}{1-2x}$.

6. (a) F (b) F (c) F (d) T (e) T (f) F (g) F (h) T (i) F (j) T

Here are some remarks and partial explanations for answers in #6:

(a) Compare #2, parts (b) and (c).

(b) For example, #1(a).

(c) $\frac{1}{x}$ is not a power series centered at 0. (Every rational function can however be expressed as a power series centered at some other point. Although we have not used power series centered at points other than 0, I would be obligated for this reason to count either answer T or F as correct. Oops.)

(d) The sequence has generating function $\frac{1+x}{1-3x-2x^2}$ where the denominator factors as $(1-\alpha x)(1-\beta x)$. Here $\alpha = 1+\sqrt{3}$ and $\beta = 1-\sqrt{3}$. The sequence has the form $a_n = A\alpha^n + B\beta^n \sim A\alpha^n$ for some nonzero constants A, B.

(e) We have a geometric series $\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$.

(f) Consider $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ and $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$.

(g) Take $a_n = b_n = a_n b_n = 1$.

(h) This was shown in class and on the related handout on counting walks in graphs.

(i) Recall that $F(x) = \frac{1}{1-x-x^2}$. This clearly does not satisfy the stated functional equation, nor is there any reason to expect it would, unless you're confusing the sequence of coefficients with the generating function itself.

(j) The sequence $1, 2, 2, 2, 2, \ldots$ has generating function $1 + 2x + 2x^2 + 2x^3 + \cdots = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$.