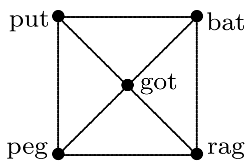
Solutions to Sample Test 1

March 2023

1. (a)

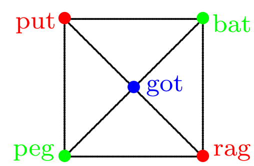


(b) Yes, Γ is planar since it is shown in (a) without any edges crossing.

(c) We have $\omega(\Gamma) = 3$. Every clique in Γ contains at most two vertices in the 'outer' 4-cycle; so including the 'center' vertex 'got', every clique has at most 3 vertices.

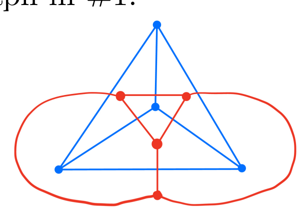
(d) We have $\alpha(\Gamma) = 2$. A coclique in Γ cannot contain the center vertex 'got', and it contains at most two vertices of the outer 4-cycle.

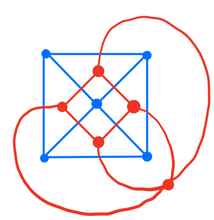
(e) The chromatic number is $\chi(\Gamma) = 3$. Here is a proper 3-coloring of the vertices of Γ. There is no proper 2-coloring of the vertices because Γ contains triangles.



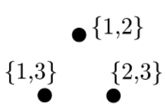
(f) We have $|\operatorname{Aut}\Gamma| = 8$. There are at least 8 automorphisms because our illustration of Γ in (a) has the full symmetry group of the square. There cannot be any more automorphisms than these, because every automorphism fixes the unique vertex of degree 4; and the remaining four vertices form a 4-cycle with only 8 automorphisms.

2. Note that any example has the same number of vertices and regions. Two examples are K_4 and the graph in #1.

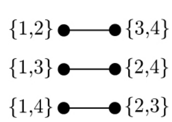




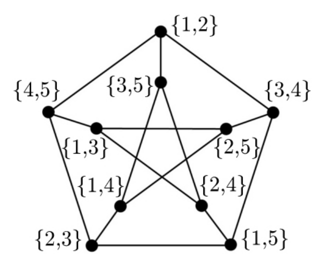
3. (a)



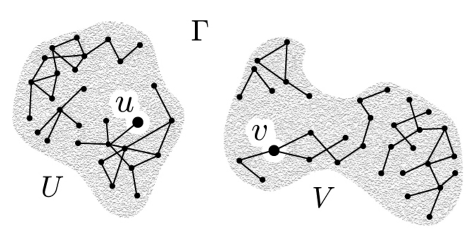
(b)



(c)



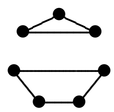
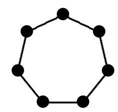
4. (a) If Γ is disconnected, then there exist vertices u, v in Γ with no path from u to v. This means that the vertex set is partitioned as $U \sqcup V$ where $u \in U$, $v \in V$ and there are no edges between U and V.



Denote by $\overline{d}(x,y)$ the distance between two vertices x, y in $\overline{\Gamma}$. If both x and y are in U, then $\overline{\Gamma}$ has edges from x to v to y, so $\overline{d}(x,y) \leqslant 2$. Similarly if both x and y are in V, then $\overline{d}(x,y) \leqslant 2$. If one of x and y is in U and the other in V, then $\overline{\Gamma}$ has an edge from x to y, so $\overline{d}(x,y) = 1$. This proves that $\overline{\Gamma}$ has diameter at most 2.

(b) No, it is impossible for both a graph and its complement to be disconnected. By (a), if Γ is disconnected, then $\overline{\Gamma}$ is connected.

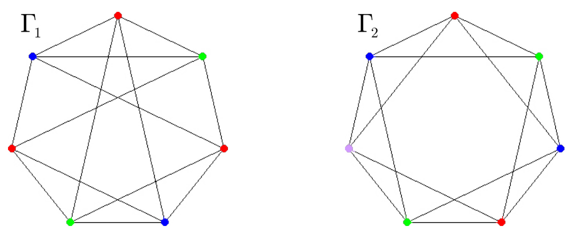
5. Note that the complementary graphs are

$\overline{\Gamma_1} =$  and $\overline{\Gamma_2} =$  .

(a) $\omega(\Gamma_1) = \alpha(\overline{\Gamma_1}) = 3$ $\quad$ $\alpha(\Gamma_1) = \omega(\overline{\Gamma_1}) = 3$ $\quad$ $\chi(\Gamma_1) = 3$

$\omega(\Gamma_2) = \alpha(\overline{\Gamma_2}) = 3$ $\quad$ $\alpha(\Gamma_2) = \omega(\overline{\Gamma_2}) = 2$ $\quad$ $\chi(\Gamma_2) = 4$

Proper colorings of the vertices (with 3 and 4 colors respectively) are



Since both graphs contain triangles, they cannot be properly colored with fewer than 3 colors. It is easy to see that Γ_2 cannot be properly colored with only 3 colors: if one colors the vertices of a triangle red, green, blue, and tries to continue with the same colors, one finds that the colors of the remaining vertices are forced, leading to a contradition with the last vertex.

(b) No. Since $\overline{\Gamma_1} \not\cong \overline{\Gamma_2}$ (see above), $\Gamma_1 \not\cong \Gamma_2$.

(c) $|\operatorname{Aut}\Gamma_1| = |\operatorname{Aut}\overline{\Gamma_1}| = 48$. The 3-cycle has 6 automorphisms, while the 4-cycle has 8 automorphisms. One therefore has $6 \times 8 = 48$ automorphisms of $\overline{\Gamma_1}$. Similarly, $|\operatorname{Aut}\Gamma_2| = |\operatorname{Aut}\overline{\Gamma_2}| = 14$ since $\overline{\Gamma_2}$ is a 7-cycle.

6. Let $\mathcal{S}$ be the set of all labelled Petersen graphs with vertex set $[10]$, and let $P \in \mathcal{S}$. We will show that $|\mathcal{S}| = 30240$. By permuting the vertex labels in all 10! ways, we can map P to any other labelled graph in $\mathcal{S}$, since they are all isomorphic. So $|\mathcal{S}| \leqslant 10! = 3{,}628{,}800$. But this is vastly overcounting, because every labelled graph in $\mathcal{S}$ is obtained 120 times (the number of automorphisms of a Petersen graph). So in fact $|\mathcal{S}| = \frac{10!}{120} = 30240$.

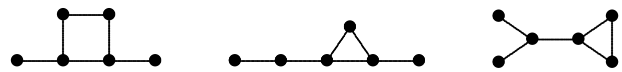
7. (a) F (b) F (c) T (d) T (e) T (f) T (g) T (h) T (i) T (j) T

Here are some remarks and partial explanations for answers in #7:

(a) H_3 has no Euler circuit since it has eight vertices of odd degree. The same argument applies to H_n whenever n is odd.

(b) If $m < n$, then every circuit in $K_{m,n}$ has length at most $2m$ since it cannot contain more than m vertices in either part of the bipartition. Such a circuit omits $n - m$ vertices in the second part.

(c) The 5-cycle is isomorphic to its complement.

(d) Some examples are 

(e) An example is . . .

(f) Every connected 2-regular graph is a cycle. And every graph is a disjoint union of its connected components.

(g) This is easily proved by induction.

(h) There are $n!$ permutations of the vertices of K_n.

(i) Number the vertices $1, 2, 3, \ldots, n$ and use colors red, blue, green, yellow. At each step $i \in \{1, 2, \ldots, n\}$, choose a color for vertex i which is different from any colors on its neighbors.

(j) If Γ is 3-regular with n vertices and e edges, then $3n = 2e$.