University of Wyoming

Math 3700—Spring 2023

Combinatorics

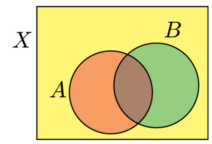
Department of Mathematics

Solutions to the Exam

May, 2023

1. (a) $26^8 = 208{,}827{,}064{,}576$. (Don't bother with these large decimal representations! I'm only working them out in case someone tries to answer that way.)

   (b) If X is the set of all passwords, A is the set of all passwords using no letters, and B is the set of passwords using no special symbols, then the number of passwords using at least one letter and at least one special symbol is $|X| - |A \cup B| = |X| - |A| - |B| + |A \cap B| = 42^8 - 16^8 - 36^8 + 10^8 = 6{,}857{,}347{,}121{,}664$. (Think of the Venn diagram.)



   (c) $P(42, 8) = 4{,}758{,}977{,}059{,}200$

   (d) $42 \cdot 41^7 = 8{,}179{,}679{,}503{,}002$. There are 42 choices for the first character, then 41 choices for each of the 7 remaining characters.

2. (a) $10! = 3{,}628{,}800$. There are only 10! permutations of the vertex set.

   (b) 25 is the maximum possible number of edges. Every bipartite graph is a subgraph of a complete bipartite graph $K_{10-m,m}$ for some $m \in \{1, 2, \ldots, 9\}$; and the number of edges in $K_{10-m,m}$ is $(10-m)m \leqslant 25$.

   (c) Again, the maximum possible number of edges is 25, and the maximum is achieved only by the bipartite graph $K_{5,5}$, by Mantel's Theorem.

   (d) The maximum number of edges is 15, achieved by the Petersen graph.

   (e) There is only 1 such graph, shown in #5(c).

   (f) Every such graph is a disjoint union of cycles of length at least 3. There are 5 such graphs, corresponding to the partitions of 10 into parts of size at least 3, namely (10), $(7, 3)$, $(6, 4)$, $(5, 5)$, $(4, 3, 3)$. (For example, a disjoint union of a 4-cycle and two 3-cycles.)

   (g) There are just 5 such graphs, namely the complements of the graphs in (f).

3. (a) This is the partition number $p(12) = 77$.

   (b) The Bell number $B_{12} = 4{,}213{,}597$.

   (c) The Stirling number of the second kind $\left\{ {12 \atop 5} \right\} = 1{,}379{,}400$.

   (d) $\binom{7+5-1}{5-1} = \binom{11}{4} = 330$. We first put one coin each envelope, then distribute the remaining 7 coins.

   (e) The number of surjections from the set of 12 coins to the set of 5 letters {A,B,C, D,E} is $\left\{ {12 \atop 5} \right\} \cdot 5! = 165{,}528{,}000$.

4. (a) $w_n = 0, 1, 5, 21$ for $n = 0, 1, 2, 3$ by direct counting. For example for $n = 3$, there are 4 paths along the vertices $(1, 1, 1, 2)$, 6 paths along $(1, 1, 2, 2)$, 9 paths along $(1, 2, 2, 2)$, and 2 paths along $(1, 2, 1, 2)$; so $w_3 = 4+6+9+2 = 21$.

(b) We are given that the sequence satisfies a linear recurrence $w_n = aw_{n-1} + bw_{n-2}$ for all $n \geqslant 2$. (But if you weren't given this information, you would know this from the fact that Γ has order 2.) The cases $n = 2, 3$ (together with the values in (a)) allow us to uniquely solve for a, b to obtain the relation $w_n = 5w_{n-1} - 4w_{n-2}$ for all $n \geqslant 2$; also $w_0 = 0$ and $w_1 = 1$. Alternatively, the recurrence relation follows from the denominator in (c).

(c) $[I - xA]^{-1} = \begin{bmatrix} 1-2x & -x \\ -2x & 1-3x \end{bmatrix}^{-1} = \frac{1}{1-5x+4x^2}\begin{bmatrix} 1-3x & x \\ 2x & 1-2x \end{bmatrix}$. The $(1, 2)$-entry gives the generating function $W(x) = W_{1,2}(x) = \frac{x}{1-5x+4x^2}$ for the sequence w_n.

(d) $\frac{x}{1-5x+4x^2} = \frac{A}{1-4x} + \frac{B}{1-x}$ so $x = (1-x)A + (1-4x)B$. Substituting $\frac{1}{4}$ and 1 gives $A = \frac{1}{3}$ and $B = -\frac{1}{3}$, so $W(x) = \frac{1/3}{1-4x} - \frac{1/3}{1-x}$.

(e) From (d),

$$W(x) \;=\; \tfrac{1}{3}\sum_{n=0}^{\infty} 4^n x^n - \tfrac{1}{3}\sum_{n=0}^{\infty} x^n \;=\; \tfrac{1}{3}\sum_{n=0}^{\infty} (4^n - 1)x^n$$

so $w_n = \frac{1}{3}(4^n - 1)$.

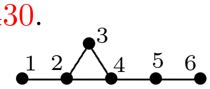
(f) $w_n \sim \frac{1}{3} \cdot 4^n$ as $n \to \infty$.

5. (a) (Remember MISSISSIPPI?) $\binom{9}{4,\,3,\,1,\,1} = 2520$.

(b) $\frac{1}{2}\binom{9}{4,\,3,\,1,\,1} = 1260$. There are 2520 words made up of four A's, three B's, one C and one D. These words tell us how to partition 9 students into groups A, B, C, D of size 4, 3, 1 and 1 respectively. But since groups C and D both have the same size, and their order doesn't matter, we divide by 2 to get 1260.

(c) $2^5 \cdot 5! = 3840$. First flip any of the edges around in $2^5 = 32$ ways, then permute the five edges in $5! = 120$ ways.

(d) $C_8 = \frac{1}{9}\binom{16}{8} = 1430$.

(e) Vertices 2,3,4 in 

can be properly colored in 6 ways, then vertex 1 in 2 ways, then vertex 5 in 2 ways, then vertex 6 in 2 ways. Altogether there are $6 \cdot 2 \cdot 2 \cdot 2 = 48$ ways.

6. (a) T (b) T (c) T (d) T (e) F (f) F (g) T (h) F (i) T (j) T

Here are some remarks and partial explanations for answers in #6:

(a) As discussed in class, $n!$ has superexponential growth. Choose an integer $N > \alpha$. We want to show that the ratio $c_n \to \infty$ where

$$c_n \;=\; \frac{n!}{\alpha^n} \;=\; \prod_{k=1}^{n} \frac{k}{\alpha} \;=\; c_{2N} \prod_{k=2N+1}^{n} \frac{k}{\alpha} \;>\; 2^{n-2N} c_{2N}$$

whenever $n > 2N$, since each of the factors $\frac{k}{\alpha} > 2$ in this range. This shows that $c_n \to \infty$ as $n \to \infty$, as required.

(b) As discussed in class, although the proof was not given. The ratio of the number of surjections $[2n] \to [n]$ to the number of injections $[n] \to [2n]$ is

$$\frac{\left\{ {2n \atop n} \right\} n!}{P(2n,n)} = \frac{\left\{ {2n \atop n} \right\}}{\binom{2n}{n}}.$$

Here you should expect that this $\to \infty$ just by a quick comparison of Stirling's triangle with Pascal's triangle. But to finish the proof, let $c_n = \left\{ {2n \atop n} \right\} / \binom{2n}{n}$ and consider the ratio $\frac{c_n}{c_{n-1}}$ as $n \to \infty$. First, $\binom{2n}{n} / \binom{2n-2}{n-1} = 2n(2n-1)/n^2 \leqslant 4$. Then using the recursive formula for Stirling numbers,

$$\left\{ {2n \atop n} \right\} = \left\{ {2n-1 \atop n} \right\} + n\left\{ {2n-1 \atop n-1} \right\} \geqslant (n-1)\left\{ {2n-2 \atop n-1} \right\} + n\left\{ {2n-2 \atop n-1} \right\} = (2n-1)\left\{ {2n-2 \atop n-1} \right\}$$

so $\frac{c_n}{c_{n-1}} \geqslant \frac{2n-1}{4} \to \infty$ as $n \to \infty$.

(c) This was discussed in class. (The proof, however, was not given as this uses delicate probabilistic arguments.)

(d) As discussed in class, $A(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials; and in fact, $g(x) = x^r - c_1 x^{r-1} - c_2 x^{r-2} - \cdots - c_{r-1}x - c_r$.

(e) As we have seen in examples, a_n typically grows exponentially.

(f) The generating function for $p(n)$ is the infinite product $\prod_{k=1}^{\infty} \frac{1}{1-x^k}$, not a rational function.

(g) As covered in class.

(h) If Γ has no 10-clique and no 10-coclique, then it has order $n < R(10,10)$, where $R(10,10)$ is a (finite) Ramsey number. There are only finitely many graphs of this size.

(i) Take the column vector with all entries equal to 1.

(j) This was covered in our discussion of the Spectral Theorem for real symmetric matrices.