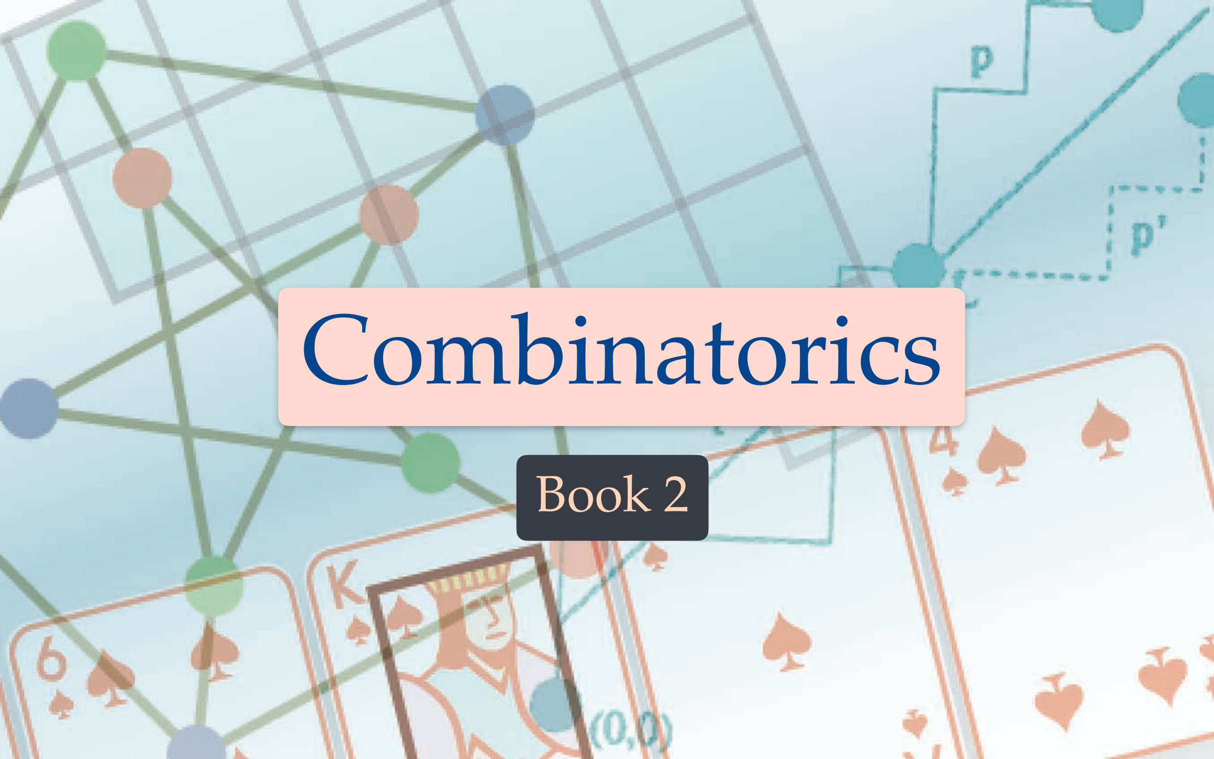
Combinatorics

Book 2



The graph $\Gamma \setminus v$ (formed by removing v and its edges from Γ) has one fewer vertex, so it can be properly colored using at most 6 colors. And since v has at most 5 neighbors in $\Gamma \setminus v$, there is a color left over which can be used to color vertex v. This gives a proper coloring of Γ using at most 6 colors (a contradiction ...)

We will improve this to show that actually 5 colors suffice to properly color every planar graph.

$\psi \ \phi \ \chi \ \pi$

A b c ... x X χ

Greek "chi"

Given a graph Γ, the chromatic number of Γ, denoted $\chi(\Gamma)$, is the smallest number of colors we can use to properly color the vertices of Γ. A proper coloring of the vertices of Γ is a coloring of the vertices such that no edge has both endpoints of the same color.

The theorem of Appel and Haken (1976) is that every planar graph Γ has $\chi(\Gamma) \leq 4$.

Note that $\chi(K_n) = n$. Here K_n is the complete graph of order n.

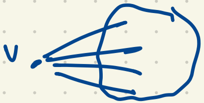
A graph Γ has $\chi(\Gamma) = 1$ iff it has vertices but no edges.

A graph Γ has $\chi(\Gamma) \leq 2$ iff Γ is bipartite iff Γ has no circuits of odd length.

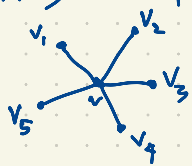
Computing $\chi(\Gamma)$ is hard in general.

Theorem If Γ is a finite planar graph then $\chi(\Gamma) \leq 5$. Proof due to Heawood.

Proof If the theorem fails then there is a smallest counterexample Γ with n vertices (so Γ is planar and every planar graph of order $n-1$ has chromatic number ≤ 5 while $\chi(\Gamma) \ge 6$). We seek a contradiction. Γ has a vertex v of degree ≤ 5. In fact $\deg v = 5$. (If $\deg v \le 4$ then $\chi(\Gamma) \le 5$, a contradiction.) Let Γ' be the graph obtained from Γ by deleting v and its five edges,

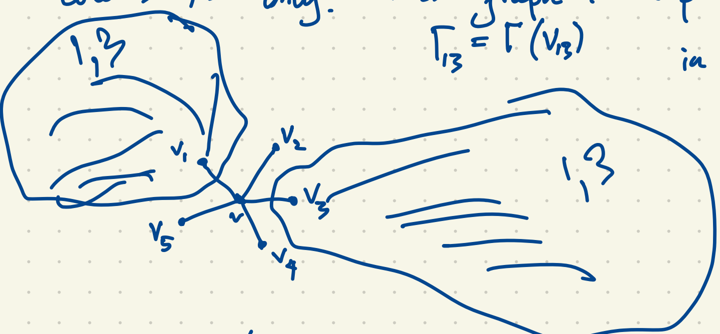


$n-1$ vertices can be properly colored using at most five colors.

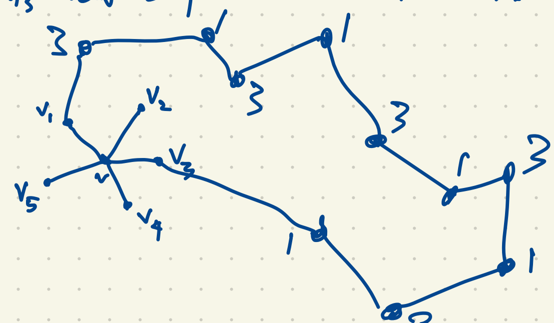


so $\chi(\Gamma') \le 5$. Say v_i has color i ($i = 1, 2, \cdots, 5$).

Consider the vertices $V_{13} \subset \{\text{vertices of } \Gamma\}$ having colors 1, 3 only. This graph is bipartite. $\Gamma_{13} = \Gamma(V_{13})$. I can assume v_1 is joined to v_3 in Γ_{13} (otherwise in part of Γ_{13}, reverse colors 1, 3 so that v_3 gets color 1. Then we are free to color v using color 3 since its neighbors are color 1, 2, 1, 4, 5).

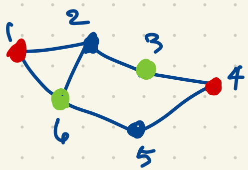


Otherwise Γ_{13} has a path from v_1 to v_3.



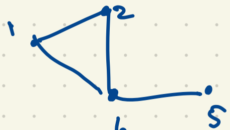
Similarly there is a path from v_2 to v_4 using only vertices of colors 2 and 4. Contradiction! $\square$

Given a graph Γ, a <u>subgraph</u> of Γ is formed by taking a subset of the edges of Γ together with all their vertices. An <u>induced subgraph</u> of Γ is formed by taking a subset of the vertices of Γ together with all their edges in Γ

$\Gamma =$ 

eg. 

is a subgraph of Γ. (not an induced subgraph of Γ)

eg. 

is an induced subgraph of Γ.

An induced subgraph of Γ is a subgraph of Γ, but not conversely.

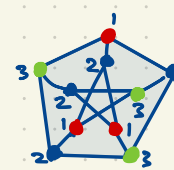
A <u>k-clique</u> in Γ is a complete subgraph of Γ, i.e. a subset of the vertices, any two of which are joined.

In Γ above, $\{1,2,6\}$ is a clique (in fact a 3-clique). The <u>clique number</u> of Γ, denoted $\omega(\Gamma)$, is the size of the largest clique in Γ. It is hard to compute $\omega(\Gamma)$.

W vs. ω
Roman Greek

<u>Theorem</u> For every graph Γ, $\chi(\Gamma) \geq \omega(K)$.

Warning: this not equality! For the Petersen graph P, $\omega(P) = 2$.

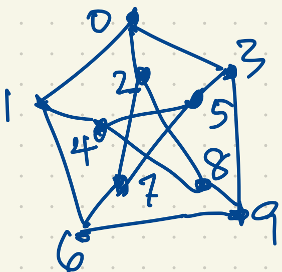


$\chi(P) = 3$ for the Petersen graph P.

Proof: The vertices in a clique of size $\omega(\Gamma)$ require $\omega(\Gamma)$ different colors.

Dual to the clique number $\omega(\Gamma)$ we have the coclique number $\alpha(\Gamma)$ which is the maximum number of vertices in Γ, no two of which are joined. (This is $\alpha(\Gamma) = \omega(\bar{\Gamma})$ where $\bar{\Gamma}$ is the complementary graph). Cocliques are also called independent sets of vertices.

Eg. $\alpha(P) = 4$. $|\chi(P)| \geq \frac{10}{4} = 2.5 \implies \chi(P) \geq 3$.



$\{1,2,3\}$ is a coclique which is not contained in any larger coclique; it is a maximal coclique.

A maximum coclique (i.e. a coclique of maximum size) is $\{1,3,7,8\}$.

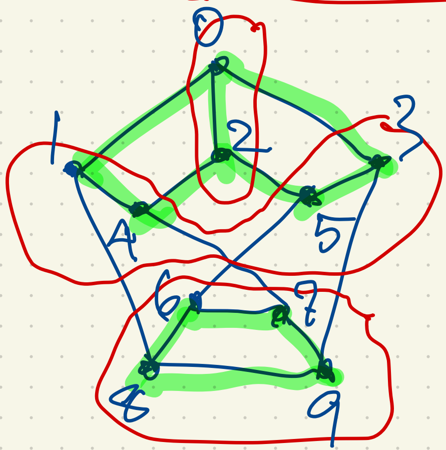
This is maximum size because P has vertex set $\{0,3,9,6,1\} \cup \{2,7,5,4,8\}$ as a union of two 5-cycles (circuits of length 5). Any set of size at least 5 vertices has either 3 on the inner 5-cycle $\{2,7,5,4,8\}$ or 3 vertices on the outer 5-cycle $\{0,3,9,6,1\}$. In either case there is an edge in that 5-cycle joining two vertices we have chosen.

Theorem $\chi(\Gamma) \geq \frac{|V|}{\alpha(\Gamma)}$ where $|V|$ = the number of vertices = the order of Γ.

Proof Let $k = \chi(\Gamma)$. Properly color the vertices $1, 2, \cdots, k$ and let V_i be the subset of vertices colored i, for $i = 1, 2, \cdots, k$. This gives a partition $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ ($V_i \cup V_j$ = union of V_i and V_j; $V_i \sqcup V_j$ = disjoint union of V_i and V_j). Each V_i is a coclique so $|V_i| \leq \alpha(\Gamma)$ so $|V| = |V_1| + |V_2| + \cdots + |V_k| \leq \underbrace{\alpha(\Gamma) + \alpha(\Gamma) + \cdots + \alpha(\Gamma)}_{k} = k\,\alpha(\Gamma)$ so $\chi(\Gamma) = k \geq \frac{|V|}{\alpha(\Gamma)}$. $\square$

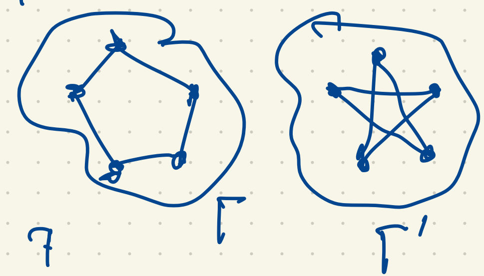
March	M	W	F
		1	3
	6	$\boxed{8}$	10
	13	15	17	Spring Break

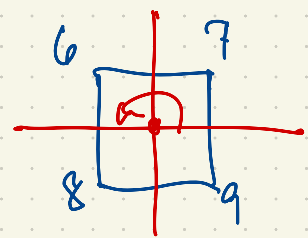
Test 1 : Wed Mar 8



$= \Gamma$

You can use nauty to test isomorphism between two graphs.





Using nauty software

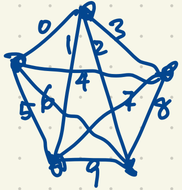
$G = \mathrm{Aut}\,\Gamma$

$G = \langle (1,3)(4,5)(6,7)(8,9),\ (0,2)(1,4)(3,5)(6,9)(7,8) \rangle$

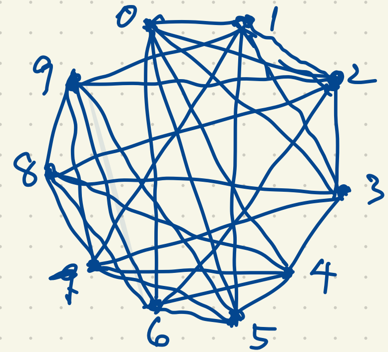
$|G| = 4$

G has 3 orbits on the vertices: $\{0,2\}, \{1,3,4,5\}, \{6,7,8,9\}$

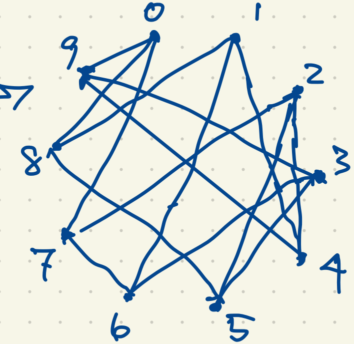
$K_5 =$



$\longrightarrow$ $K_5' $

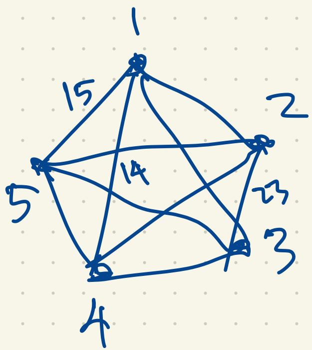


$\xrightarrow{\overline{K_5'}}$

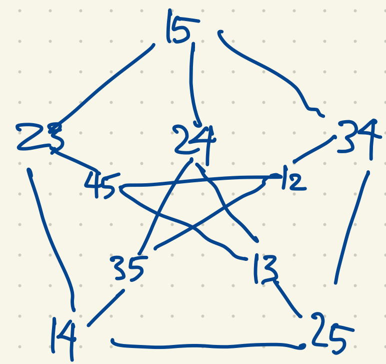


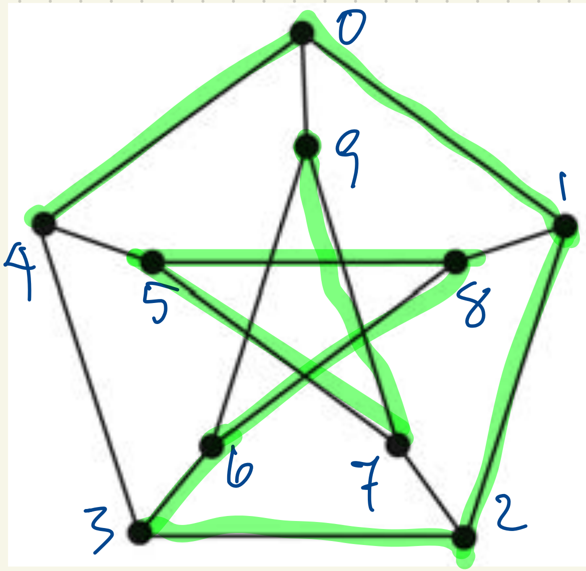
$|\mathrm{Aut}\ \overline{K_5'}| = 120$

$K_5 =$



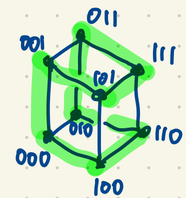
$\overline{K_5'} =$

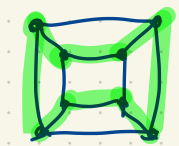




The Petersen graph P has a Hamilton path

$(0,1,2,3,6,8,5,7,9)$ (a path touching each vertex exactly once) but no Hamilton circuit (ending at the same vertex where it started).

The Hamming cube H_3 = 

= 

does have a Hamilton circuit.

000
100
110
010
011
111
101
001
000

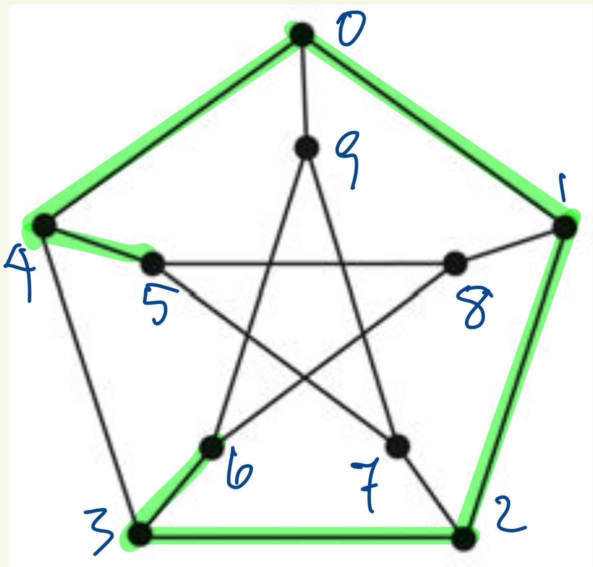
"Gray code"

Every Hamming graph H_n $(n \geq 2)$ has a Hamilton circuit.

A graph having a Hamilton circuit is called <u>Hamiltonian</u>.

Looking for Hamilton paths or circuits is known to be difficult in general.

Testing whether a giving graph Γ is Hamiltonian is NP-complete.

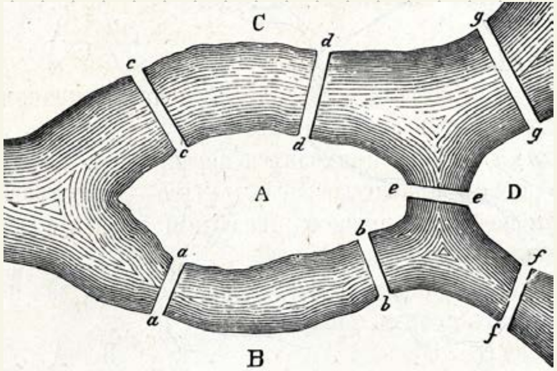


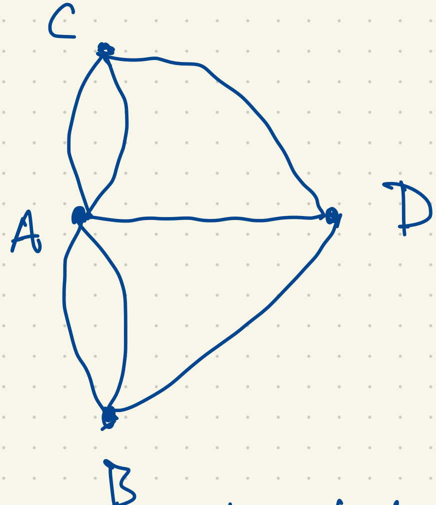
Theorem: The Petersen graph P is not Hamiltonian, i.e. it does not have a Hamilton circuit/cycle.

Proof Suppose P has a Hamilton circuit. Without loss of generality this circuit contains the path (4, 0, 1, 2). (This is because P has 120 automorphisms mapping any such path of length 3 to any other.) The Hamilton circuit uses two of the edges from vertex 3, so it uses either {3,4} or {2,3}; so without loss of generality, it uses the edge {2,3}. We cannot use the edge {3,4} as this would complete the circuit without passing through all vertices; so we must use the edges {3,6} and {4,5}. To continue the circuit from vertex 6, we have two choices: proceed through vertex 8 or vertex 9. Neither of these choices leads to a Hamilton circuit. This is a contradiction. □

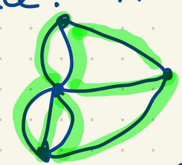
Euler paths and circuits

The Seven bridges of Königsberg





An Euler trail is a trail (repeating vertices but not edges) which uses each edge exactly once. An Euler circuit is an Euler trail that returns to its starting point.

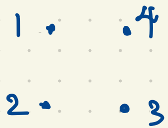


This graph has an Euler trail. In order to have an Euler trail, a graph must have either 0 or 2 vertices of odd degree. When there are no vertices of odd degree, we have an Euler circuit.

Theorem (Euler) A graph has an Euler trail iff it is connected and it has either 0 or 2 vertices of odd degree. In the case every vertex has even degree, we have an Euler circuit/cycle.

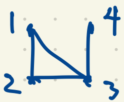
We sometimes speak of labelled graphs and unlabelled graphs.

Eg. on the vertex set $\{1,2,3,4\}$, there are $2^6 = 64$ labelled graphs

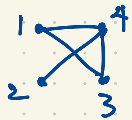


$\binom{4}{2} = 6$ pairs of vertices.

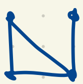
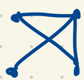
There are $\binom{n}{2}$ labelled graphs on n vertices.
But many ~~of~~ them are isomorphic.



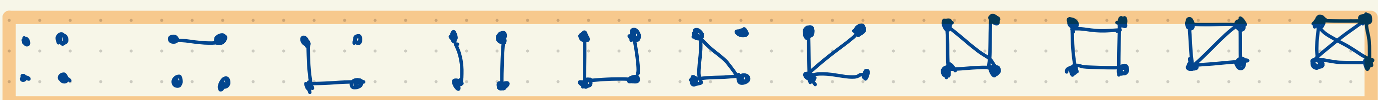
$\neq$



These are different labelled graphs but they are isomorphic.

 $\stackrel{?}{=}$ 

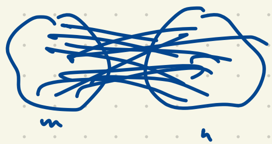
As unlabelled graphs they are isomorphic, hence the same graph.



There are 11 unlabelled graphs of order 4 i.e. 11 isomorphism types of graphs of order 4, i.e. 11 graphs of order 4 up to isomorphism.

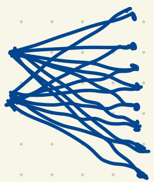
The Petersen graph has girth 5 (the shortest cycle has length 5). It has 15 edges.
For a graph on 10 vertices, 15 edges is the maximum possible for girth 5.
For a graph on 10 vertices without triangles (i.e. girth ≥ 4), what is the maximum possible number of edges?

Recall: $K_{m,n}$ =



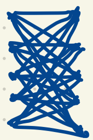
$K_{m,n}$ has mn edges. $K_{m,n}$ is bipartite so it has no cycles of odd length. In particular it has no triangles.

$K_{2,8}$ has 16 edges



$K_{2,8}$

16 edges



$K_{5,5}$

25 edges
girth 4
(no triangles)

<u>Theorem</u> (Mantel 1907)
If Γ is a graph of order n with no triangles (i.e. its girth is at least 4) then Γ has at most $\frac{n^2}{4}$ edges.

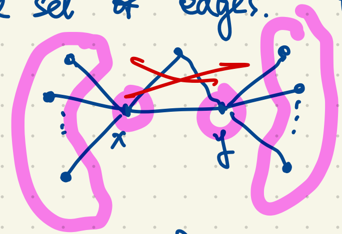
If n is even then $K_{\frac{n}{2},\frac{n}{2}}$ attains the upper bound of $\frac{n^2}{4}$ edges. What if n is odd?

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even, for } K_{\frac{n}{2},\frac{n}{2}} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd, for } K_{\frac{n+1}{2},\frac{n-1}{2}} \end{cases}$$

On 9 vertices, any graph without triangles has at most <u>20</u> edges.

$K_{4,5}$

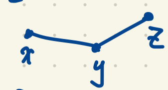
Proof Let Γ be a graph of order n with no triangles, $\Gamma = (V, E)$. (V is the set of vertices, E is the set of edges.) For every edge $\{x,y\} \in E$, $d(x) + d(y) \le n$.

$d(x)-1$  $d(y)-1$

$$d(x)-\cancel{1} + \cancel{1} + \cancel{1} + d(y) - \cancel{1} \le n$$

Add the inequality $d(x)+d(y) \le n$ over all edges $\{x,y\} \in E$ to get $\sum_{\{x,y\}\in E} (d(x)+d(y)) \le ne$.

Next, count the number of triples of vertices (x,y,z) with $x \sim y \sim z$.

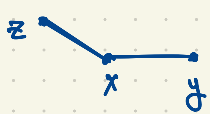


There are n choices for $y \in V$ and $d(y)$ choices for x, $d(y)$ choices for z, so $d(y)^2$ choices for x and z (given y). The total number of walks of length 2 is $\sum_{y \in V} d(y)^2$.

On the other hand, there are $e = |E|$ edges in Γ. For the edge $\{x,y\} \in E$, how many walks of length 2 contain this edge? $d(x)+d(y)$ choices of walk of length 2 in which we include a step from x to y

The total number of walks of length 2 is

$$\sum_{\{x,y\}\in E} (d(x)+d(y)).$$



$d(x)$ choices for z
(given the edge $\{x,y\}$)

or

$d(y)$ choices for z
(given the edge $\{x,y\}$)

Therefore

$$\sum_{x\in V} d(x)^2 = \sum_{\{x,y\}\in E} \underbrace{(d(x)+d(y))}_{\le n} \le en$$

If $\sum_{x\in V} d(x) = 2e$, what does this tell us about $\sum_{x\in V} d(x)^2$?

Use the Cauchy-Schwarz inequality.

$$(\underbrace{d(1)+\cdots+d(n)}_{2e})^2 \le n(d(1)^2+\cdots+d(n)^2)$$

$$4e^2 \le n\sum_{x\in V} d(x)^2 = n\sum_{\{x,y\}\in E}(d(x)+d(y)) \le n\cdot ne = n^2 e$$

So $e \le \frac{n^2}{4}$.

Cauchy-Schwarz Inequality

Given two vectors

$a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$

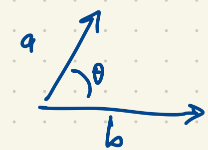
$b = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$,

$$|a\cdot b| \le \|a\|\cdot\|b\|$$

where $a\cdot b = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

$$a\cdot b = \|a\|\,\|b\|\cos\theta$$



$|\cos\theta| \le 1$.

Special case: $b = (1, 1, 1, \cdots, 1)$

$\|b\| = \sqrt{\underbrace{1^2+1^2+\cdots+1^2}_{n}} = \sqrt{n}$

$a\cdot b = a_1 + a_2 + \cdots + a_n$

$|a\cdot b| \le \|a\|\cdot\sqrt{n}$

$$(a_1+\cdots+a_n)^2 \le n(a_1^2+\cdots+a_n^2)$$

Proof of Cauchy-Schwarz: Fix $a, b \in \mathbb{R}^n$. Consider

$$f(t) = \|a - tb\|^2 = (a-tb)\cdot(a-tb) = a\cdot a - tb\cdot a - ta\cdot b + t^2\, b\cdot b = \|a\|^2 - 2t(a\cdot b) + t^2\|b\|^2.$$

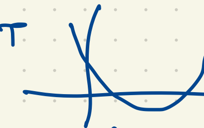
Of course $f(t) \geq 0$ for all t.

disc < 0

or

disc $= 0$

NOT



disc > 0

The discriminant

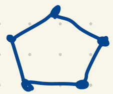
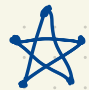
$$(-2a\cdot b)^2 - 4\|a\|^2\|b\|^2 \leq 0$$

$$4(a\cdot b)^2 \leq 4\|a\|^2\|b\|^2$$

$$(a\cdot b)^2 \leq \|a\|^2\|b\|^2.$$

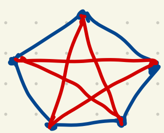
The disc. of at^2+bt+c is b^2-4ac

Show me a graph Γ of order 5 such that neither Γ nor $\bar{\Gamma}$ has a triangle.
(i.e. both Γ and $\bar{\Gamma}$ have girth ≥ 4). Recall: $\bar{\Gamma}$ is the complement of Γ.

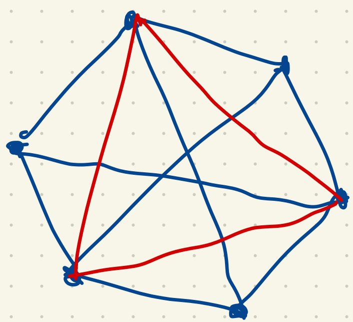
$\Gamma =$  $\bar{\Gamma} =$  $\cong \Gamma$ (a 5-cycle). Both Γ and $\bar{\Gamma}$ have girth 5.

Show me a graph Γ of order 6 such that neither Γ nor $\bar{\Gamma}$ have a triangle.
There is no such graph. Why not?

Color the edges of K_5 with 2 colors red, blue. To avoid a monochromatic triangle (all red or all blue):

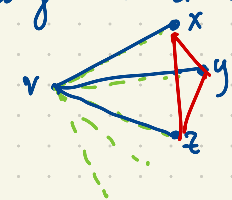


K_6:



Theorem If we color the edges of K_6 red and blue, then there is either a red triangle or a blue triangle.

Proof Consider a vertex v.



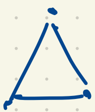
There are five edges from v so by the Pigeon hole Principle, at least three of them are the same color, say $\{v,x\}$, $\{v,y\}$, $\{v,z\}$ are blue.

Now x,y,z form a triangle.
If any edge of this triangle is blue then together with the edges to v we have a blue triangle. Otherwise all edges of the triangle $\{x,y,z\}$ are red. $\square$

Theorem (Ramsey) Let r, s be positive integers. There is a number $R(r,s)$ such that for all $n \geq R(r,s)$, every 2-coloring (red, blue) of the edges of K_n has either a red r-clique or a blue s-clique. For $n < R(r,s)$, there exists a coloring of the edges of K_n with 2 colors (red, blue) having no red r-clique and no blue s-clique.

$R(3,3) = 6.$

$R(2,3) = 3$



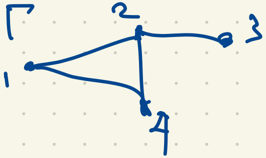
$43 \leq R(5,5) \leq 48$

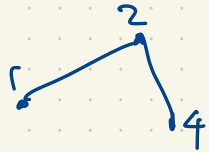
$R(r,s)$ values are Ramsey numbers.

(Extremal graph theory)

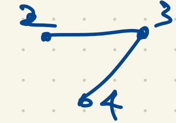
Values / known bounding ranges for Ramsey numbers $R(r, s)$ (sequence A212954 in the OEIS)

r \ s	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–42
4				**18**	25[9]	36–40	49–58	59[13]–79	73–106	92–136
5					**43–48**	58–85	80–133	101–194	133–282	149[13]–381
6						**102–161**	115[13]–273	134[13]–427	183–656	204–949
7							**205–497**	219–840	252–1379	292–2134
8								**282–1532**	329–2683	343–4432
9									**565–6588**	581–12677
10										**798–23556**

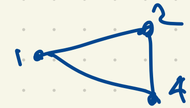




is a subgraph of Γ



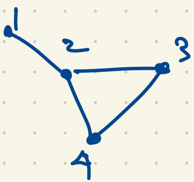
is not a subgraph.



is an induced subgraph.

$$\underbrace{\text{Graph Theory} \longleftrightarrow \text{Linear Algebra}}_{\text{Matroid Theory}}$$

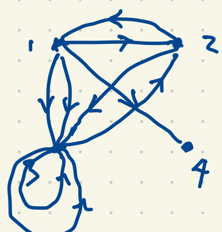
The adjacency matrix of a graph Γ with vertices $1, 2, 3, \cdots, n$ is the $n \times n$ matrix whose (i,j)-entry is the number of edges from vertex i to vertex j. Eg.

Γ =  has adjacency matrix

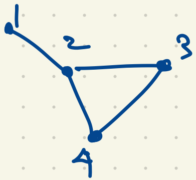
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = A$$

Symmetric $(0,1)$-matrix with "zero diagonal" (corresponding to an undirected graph with no loops or multiple edges)

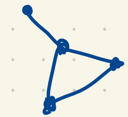
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



directed graph with loops and multiple edges

$\Gamma =$ 

has adjacency matrix
$$\begin{array}{c|c} & \begin{matrix}1&2&3&4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}0&1&0&0\\1&0&1&1\\0&1&0&1\\0&1&1&0\end{bmatrix}\end{array} = A$$

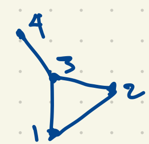
The unlabelled graph  has several choices of adjacency matrix.

$$A^2 = \begin{bmatrix}0&1&0&0\\1&0&1&1\\0&1&0&1\\0&1&1&0\end{bmatrix}\begin{bmatrix}0&1&0&0\\1&0&1&1\\0&1&0&1\\0&1&1&0\end{bmatrix} = \begin{array}{c|c} & \begin{matrix}1&2&3&4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}1&0&1&1\\0&3&1&1\\1&1&2&1\\1&1&1&2\end{bmatrix}\end{array}$$

The (i,j)-entry of A^2 is the number of walks of length 2 from vertex i to vertex j.
(A walk is allowed to repeat edges or vertices, unlike in a path or a trail.)

$$A^3 = \underbrace{\begin{bmatrix}1&0&1&1\\0&3&1&1\\1&1&2&1\\1&1&1&2\end{bmatrix}}_{A^2}\underbrace{\begin{bmatrix}0&1&0&0\\1&0&1&1\\0&1&0&1\\0&1&1&0\end{bmatrix}}_{A} = \begin{bmatrix}0&3&1&1\\3&2&4&4\\1&4&2&3\\1&4&3&2\end{bmatrix}$$

The (i,j)-entry of A^m is the number of walks of length m from vertex i to vertex j in Γ.

 has adjacency matrix $\begin{bmatrix}0&1&1&0\\1&0&1&0\\1&1&0&1\\0&0&1&0\end{bmatrix}$

$$A^0 = I = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \qquad A^1 = A$$

$$J = \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}, \quad j = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

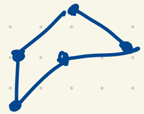
If Γ is a k-regular graph then $Aj = kj$, $AJ = kJ$
i.e. j is an eigenvector of A with eigenvalue k.

$$Aj = \begin{bmatrix}0&1&0&0\\1&0&1&1\\0&1&0&1\\0&1&1&0\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\3\\2\\2\end{bmatrix} \text{ degree sequence}$$

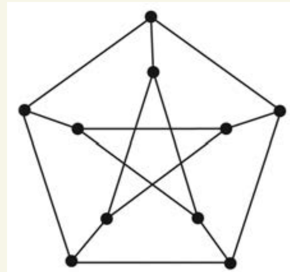
$$AJ = \begin{bmatrix}1&1&1&1\\3&3&3&3\\2&2&2&2\\2&2&2&2\end{bmatrix}$$

Which graphs are connected regular with diameter 2 and girth 5? Any such graph has order 5, 10, 50 or 3250.

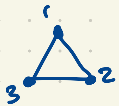
5-cycle



Petersen graph



Look at $K_3 =$



with adjacency matrix $A = \begin{array}{c|ccc} i \backslash j & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left[\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}\right] & \\ 3 & & & \end{array}$. The number of walks of length k from vertex i to vertex j is the (i,j)-entry of A^k.

How many walks of length 10 are there from vertex 2 to vertex 3 in K_3? 341.

$$A^{10} = \begin{bmatrix} 342 & 341 & 341 \\ 341 & 342 & 341 \\ 341 & 341 & 342 \end{bmatrix}$$

Since A is a real symmetric matrix, it is diagonalizable i.e. there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A. One obvious eigenvector $\underline{u}_1 = \underline{j} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: $A\underline{j} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\underline{j}$

since K_3 is 2-regular. Another eigenvector $\underline{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$: $A\underline{u}_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -\underline{u}_2$

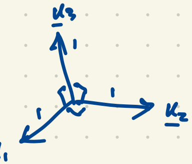
$\underline{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $A\underline{u}_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = -\underline{u}_3$

It is customary to normalize the eigenvectors as

$$\underline{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \underline{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \underline{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

so that $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

Orthonormal means $\underline{u}_i \cdot \underline{u}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$



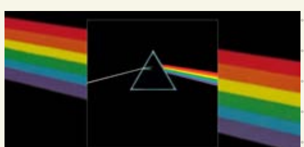
$A\underline{u}_i = \lambda_i \underline{u}_i \qquad \lambda_1 = 2, \quad \lambda_2 = \lambda_3 = -1.$

A has diagonal form $D = \begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$. $\quad D^{10} = \begin{bmatrix}1024&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$A = UDU^{-1}$ i.e. $U^{-1}AU = D$ where $U = \begin{bmatrix}\underline{u}_1 & \underline{u}_2 & \underline{u}_3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{bmatrix}$

$A^{10} = (UD\cancel{U^{-1}})(\cancel{U}D\cancel{U^{-1}})(\cancel{U}D\cancel{U^{-1}})\cdots(\cancel{U}DU^{-1}) = UD^{10}U^{-1} = \begin{bmatrix}342&341&341\\341&342&341\\341&341&342\end{bmatrix}$

$A = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix} \xrightarrow{\text{diagonalize}} D = \begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$



The list of eigenvalues $2, -1, -1$ is the spectrum of A (or of D)

$(x-2)(x^2+2x+1)$
$= x^3 - 0x^2 - 3x - 2$

Check: $\det A = 2 \qquad \operatorname{tr} A = 0$
$\det D = 2 \qquad \operatorname{tr} D = 0$

A, D are similar so they have the same trace, determinant and characteristic polynomial

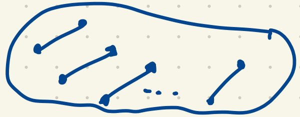
$\det(xI - A) = \det(xI - D) = (x-2)(x+1)^2$
$= x^3 - 0x^2 - 3x - 2$

$\begin{bmatrix}1&1\\0&1\end{bmatrix}$ has eigenvector $\begin{bmatrix}1\\0\end{bmatrix}$ but no basis of eigenvectors.

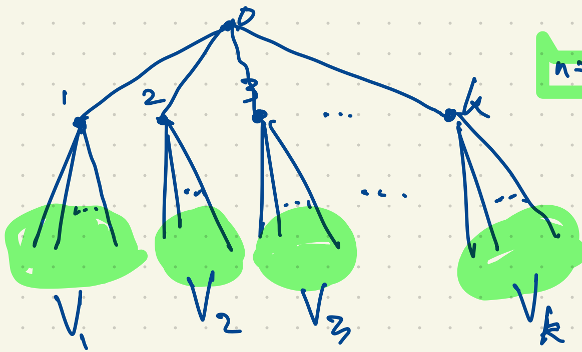
$\begin{bmatrix}3&1\\0&2\end{bmatrix}$ has a basis of eigenvectors $\begin{bmatrix}1\\0\end{bmatrix}$ (with eigenvalue 3) $\begin{bmatrix}1\\-1\end{bmatrix}$ (with eigenvalue 2) but they aren't perpendicular/orthogonal.

A **Moore graph** is a k-regular graph of diameter 2 and girth 5. What are the examples?

$k \geq 2$



1-regular graphs are not connected unless $n = 2$.

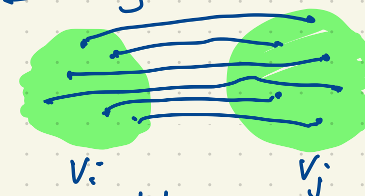


$$\boxed{n = k+1 + k(k-1) = k^2+1}$$

$|V_1| = k-1$

$|V_i| = k-1$

Between V_i and V_j $(i \neq j)$ the edges form a perfect matching



$\left((k-1)!\right)^{\frac{k(k-1)}{2}}$ ways to choose these perfect matchings.

If you can do this without creating any 3-cycles or 4-cycles then you have a Moore graph.

Now let A be the adjacency matrix of the Moore graph of degree k, $n = k^2+1$. A is $n \times n$, $I = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n}$

$J = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n}$ $\quad \underline{j} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$ $\quad A\underline{j} = k\underline{j}$ $\quad AJ = kJ$, $JA = kJ$

$A^T = A$

$(AJ)^T = (kJ)^T$

$JA = J^T A^T = kJ$

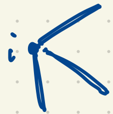
A, J, I all commute with each other.

$J - I - A$ is the adjacency matrix of the complementary graph.

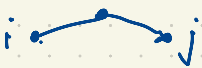
$\boxed{A^2 = kI + (J - I - A)}$

why? The (i,j)-entry of A^2 is the number of walks of length 2 from vertex i to vertex j.

Case $i=j$: A^2 has k in the diagonal entry (i,i).



Case $i \sim j$ . Here the (i,j)-entry of A^2 is 0. Same on the right hand side.

Case $i \not\sim j$, $i \neq j$  Here the (i,j)-entry on both sides is 1.

Since A is a real symmetric $n \times n$ matrix, it has n eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ with corresponding eigenvectors $\underline{u}_1, \cdots, \underline{u}_n$ i.e. $A\underline{u}_i = \lambda_i \underline{u}_i$. Some of the eigenvalues will be repeated. One of the eigenvalues is k since $A\underline{j} = k\underline{j}$, with eigenvector $\underline{j} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ or $\frac{1}{\sqrt{n}}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. What are the other eigenvalues and eigenvectors?

Say $\underline{u}$ is an eigenvector different from $\frac{1}{\sqrt{n}}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, with corresponding eigenvalue λ so $A\underline{u} = \lambda\underline{u}$.

So $\underline{u}$ is perpendicular (orthogonal) to $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ i.e. $J\underline{u} = \underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$A^2\underline{u} = [kI + (J - I - A)]\underline{u}$$
$$\lambda^2\underline{u} = k\underline{u} - \underline{u} - \lambda\underline{u}$$

$$\lambda^2 = k - 1 - \lambda$$
$$\lambda = k - 1 - \lambda^2 < k$$

$$A^2\underline{u} = AA\underline{u} = A\lambda\underline{u} = \lambda A\underline{u} = \lambda^2\underline{u}$$

So $\lambda_1 = k$ and $\lambda_2, \cdots, \lambda_n$ are less than k.

Since we have $n-1$ remaining eigenvalues $\lambda_2, \cdots, \lambda_n$ all of which are roots of $\lambda^2+\lambda-(k-1)=0$, some of the eigenvalues must be repeated. They are

$$\tfrac{1}{2}\left(-1 \pm \sqrt{1+4(k-1)}\right) = \tfrac{1}{2}\left(-1 \pm \sqrt{4k-3}\right)$$

$$\lambda = \tfrac{1}{2}\left(-1+\sqrt{4k-3}\right)$$
$$\mu = \tfrac{1}{2}\left(-1-\sqrt{4k-3}\right)$$

So the spectrum of A is $k, \underbrace{\lambda, \lambda, \cdots, \lambda}_{m}, \underbrace{\mu, \mu, \cdots, \mu}_{n-1-m}$. Since $\operatorname{tr} A = 0$,

$$k + m\lambda + (n-1-m)\mu = 0.$$

This allows us to solve for m. $\qquad n = k^2+1$

$$2k + m\left(-1+\sqrt{4k-3}\right) + (n-1-m)\cdot\left(-1-\sqrt{4k-3}\right) = 0$$

$$2k + -\underbrace{(n-1)}_{k^2} + \underbrace{(2m-n+1)}_{2m-k^2}\sqrt{4k-3} = 0$$

$$(2m-k^2)\sqrt{4k-3} = k(k-2). \quad (*)$$

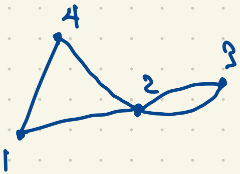
If $k=2$ we get $\lambda = \frac{1}{2}(-1+\sqrt{5})$, $\mu = \frac{1}{2}(-1-\sqrt{5})$ and the spectrum is $2, \underbrace{\lambda, \lambda}_{m=2}, \mu, \mu$

Otherwise $k>2$ so $4k-3$ must be an integer square i.e. $4k-3 = d^2$ where d is an integer, $d>0$. $k = \frac{d^2+3}{4}$. Substitute into $(*)$ to get

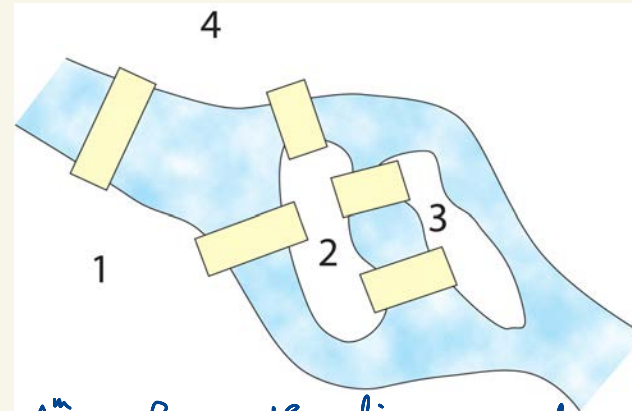
$$(d^4+d^3+6d^2-2d+9-32m)d = 15. \implies d = 1, 3, 5 \text{ or } 15. \implies k = \cancel{1}, 3, 7 \text{ or } 57.$$

Counting walks in graphs

How many ways can we walk from land mass i to land mass j by crossing m bridges?



Γ



This number $w_m(i,j)$, equals the number of walks of length m in Γ. It is also the (i,j)-entry of A^m where the adjacency matrix of Γ is

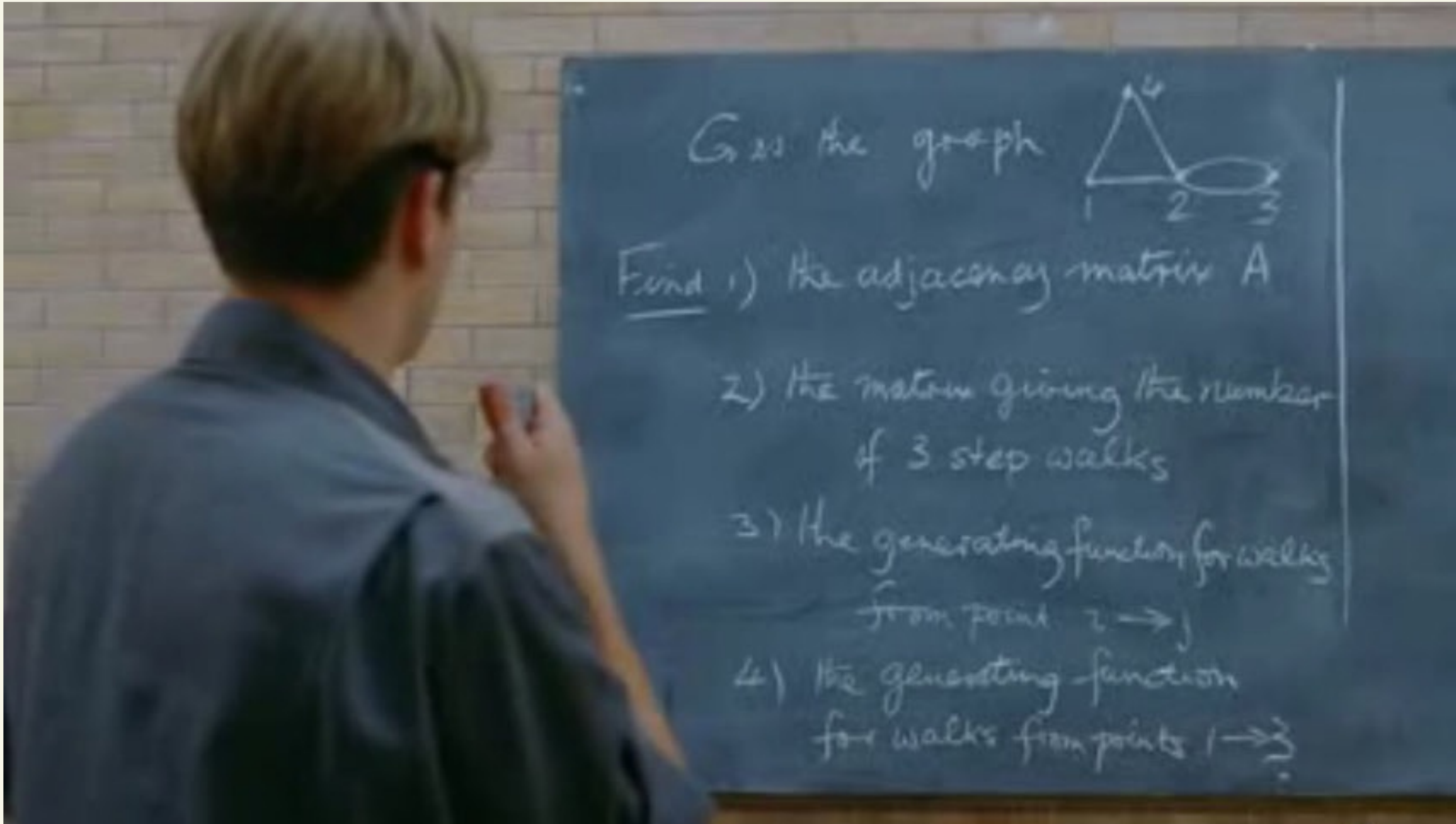
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Rather than working out A^m in each case, we want a "formula" for $w_m(i,j)$.
We are able to give a closed formula for

$$W_{ij}(x) = \sum_{m=0}^{\infty} w_m(i,j)x^m = w_0(i,j) + w_1(i,j)x + w_2(i,j)x^2 + w_3(i,j)x^3 + \cdots$$

which is the <u>generating function</u> for the sequence $w_0(i,j), w_1(i,j), w_2(i,j), \ldots$
$W_{ij}(x)$ is the (i,j)-entry of the matrix

$$W(x) = \sum_{m=0}^{\infty} A^m x^m = I + Ax + A^2x^2 + A^3x^3 + A^4x^4 + \cdots$$


G is the graph
4
1 2 3
Find 1) the adjacency matrix A
2) the matrix giving the number of 3 step walks
3) the generating function for walks from point i → j
4) the generating function for walks from points 1 → 3

Recall: A geometric series $1+r+r^2+r^3+\cdots = \frac{1}{1-r}$ if $|r|<1$.

But in a formal power series, convergence is a non-issue because the terms are purely symbolic, not numbers.

Check: cross-multiply. $(1-r)(1+r+r^2+r^3+\cdots) = 1\cancel{-r}\cancel{+r}\cancel{-r^2}\cancel{+r^2}\cancel{-r^3}\cancel{+r^3}\cancel{-r^4}\cancel{+r^4}-\cdots = 1$.

$$W(x) = I + Ax + A^2x^2 + A^3x^3 + \cdots = (I-Ax)^{-1}$$

Check: $(I-Ax)(I+Ax+A^2x^2+A^3x^3+\cdots) = I\cancel{-Ax}+\cancel{Ax}-\cancel{A^2x^2}+\cancel{A^2x^2}-\cancel{A^3x^3}+\cancel{A^3x^3}-\cdots$

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$I - Ax = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & -x & 0 & -x \\ -x & 1 & -2x & -x \\ 0 & -2x & 1 & 0 \\ -x & -x & 0 & 1 \end{bmatrix}$$

$(I-Ax)^{-1} = \frac{1}{d(x)}\operatorname{adj}(I-Ax)$ where $d(x) = \det(I-Ax)$ and $\operatorname{adj}(I-Ax)$ is the classical adjoint of $I-Ax$.

We won't work out 4x4 inverses of matrices with polynomial entries by hand.

$$(I-Ax)^{-1} = W(x) = \frac{1}{d(x)} \begin{bmatrix} 1-5x^2 & x(1+x) & 2x^2(1+x) & \cdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where $d(x) = 4x^4 + \cdots$

The (1,3)-entry is $W_{13}(x) = \frac{2x^2(1+x)}{d(x)} = \frac{2x^2}{1-x-6x^2+4x^3} = 2x^2+2x^3+14x^4+18x^5+94x^6+146x^7+638x^8+\cdots$

A bit is a '0' or a '1', expressing a binary choice.
A bitstring (binary string) is a string of bits eg. '01011' is a bitstring of length 5. The number of bitstrings of length n is 2^n.
Bitstrings of length 0: ''
Bitstrings of length 1 are bits i.e. '0', '1'.
There are 2^n subsets of any n-set (e.g. $[n] = \{1,2,3,\dots,n\}$ is the standard n-set; but any set of n elements is an n-set.) Every subset of $[n]$ is represented by a bitstring of length n, its characteristic bitstring e.g. in $[5]$, $\{2,3,5\}$ is represented by '01101'.

A bitstring of length n is 11-free if it has no two consecutive 1's, e.g. 01001 is 11-free but 01101 is not. Let a_n be the number of 11-free bitstrings of length n.
How do we compute a_n?

n	11-free bitstrings of length n	a_n
0	''	1
1	0, 1	2
2	00, 10, 01	3
3	000, 100, 010, 001, 101	5
4	0000, 1000, 0100, 0010, 0001, 1010, 1001, 0101	8
5		13

Guess: $a_n = F_{n+1}$ where $F_0, F_1, F_2, F_3, F_4, \dots$ is the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$ defined recursively by

$$F_n = \begin{cases} 1, & \text{if } n = 0 \text{ or } 1; \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2. \end{cases}$$

linear recurrence of depth 2.

Any sequence $a_0, a_1, a_2, \cdots$ satisfying $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ ($c_1, \cdots, c_k$ are explicitly known constants) is a recursively defined sequence given by a <u>linear recurrence of depth k</u>. Together with the initial conditions (values of $a_0, a_1, \cdots, a_{k-1}$) the sequence is uniquely defined.

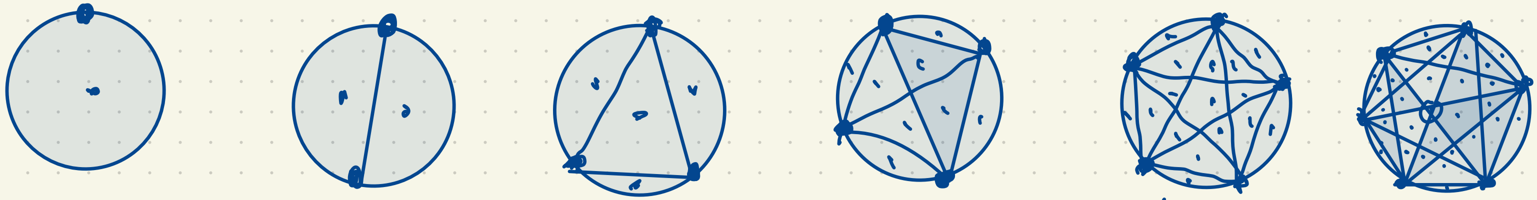
<u>Theorem</u> a_n is defined recursively by $a_n = \begin{cases} 1, & \text{if } n=0; \\ 2, & \text{if } n=1; \\ a_{n-1}+a_{n-2}, & \text{if } n \geq 2. \end{cases}$

n	11-free bitstrings of length n	a_n	b_n	c_n
0	''	1	0	1
1	0, 1	2	1	1
2	00, 10, 01	3	1	2
3	000, 100, 010, 001, 101	5	2	3
4	0000, 1000, 0100, 0010, 0001, 1010, 1001, 0101	8	3	5
5		13		

<u>Proof</u> Let b_n be the number of 11-free bitstrings of length n ending in '1';
" c_n " — — — — — — — — — not ending in '1'.

Clearly $a_n = b_n + c_n$ and $b_{n+1} = c_n$ so $c_{n+1} = c_{n-1} + c_n$

$c_{n+1} = a_n$ and $a_n = a_{n-1} + a_{n-1}$ $\square$



$2^0=1$ $\quad 2^1=2$ $\quad 2^2=4$ $\quad 2^3=8$ $\quad 2^4=16$ $\quad 31$ (or 30)

Dividing a disk into regions formed by all $\binom{n}{2}$ chords whose endpoints are n given points on the circumference, gives 2^{n-1} regions? $\boxed{\text{NO!}}$

We want a closed formula for a_n from which we can compute a_n directly for any n, and discern the rate of growth of the sequence.

For this, introduce the generating function of the sequence $a_0, a_1, a_2, \dots$

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= 1 + 2x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n$$

$$= 1 + 2x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$= 1 + 2x + x \underbrace{\sum_{n=2}^{\infty} a_{n-1} x^{n-1}}_{F(x)-1} + x^2 \underbrace{\sum_{n=2}^{\infty} a_{n-2} x^{n-2}}_{F(x) = \sum_{m=0}^{\infty} a_m x^m}$$

$m = n-2$
$n = m+2$

$$= 1 + 2x + x(F(x) - 1) + x^2 F(x)$$

$$(1 - x - x^2) F(x) = 1 + x$$

$$F(x) = \frac{1+x}{1-x-x^2}$$

There are maybe 5 ways you can expand this as a series.

(i) Compute $F'(x) = \dfrac{(1-x-x^2)\cdot 1 - (1+x)(-1-2x)}{(1-x-x^2)^2} = \dfrac{2 + 2x + x^2}{(1-x-x^2)^2}$

$F''(x),\ F'''(x),\ F^{(4)}(x),\ F^{(5)}(x),\ \dots$

Taylor series centred at $x=0$

$$F(x) = F(0) + F'(0)x + \frac{F''(0)}{2}x^2 + \frac{F'''(0)}{6}x^3 + \frac{F^{(4)}(0)}{24}x^4 + \cdots$$

$$= 1 + 2x + 3x^2 + 5x^3 + \cdots$$

Second way: Use the geometric series $\frac{1}{1-u} = 1+u+u^2+u^3+u^4+\cdots$

$$F(x) = \frac{1+x}{1-(x+x^2)} = (1+x)\left[1+(x+x^2)+(x+x^2)^2+(x+x^2)^3+(x+x^2)^4+\cdots\right]$$

$$= (1+x)\left[1+(x+x^2)+(x^2+2x^3+x^4)+(x^3+3x^4+3x^5+x^6)+(x^4+4x^5+6x^6+4x^7+x^8)+\cdots\right]$$

$$= (1+x)\left[1+x+2x^2+3x^3+5x^4+\cdots\right]$$

$$= 1+2x+3x^2+5x^3+8x^4+13x^5+\cdots$$

Third way:

$$\frac{1+x}{1-x-x^2} = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$$

$$1+x = (1-x-x^2)(1+2x+3x^2+a_3x^3+\cdots) \qquad a_0=1$$

$$1 = 1+a_1 \implies a_1=2$$

$$0 = -1-2+a_2 \implies a_2=3$$

Note: The denominator of $F(x)$ gives us the recurrence formula $a_n = a_{n-1}+a_{n-2}$.
The numerator of $F(x)$ gives us the intial conditions.

Fourth method: Decompose $F(x) = \frac{1+x}{1-x-x^2}$ using partial fractions.

Factor the denominator $1-x-x^2 = (1-\alpha x)(1-\beta x)$

Note: The factors $1-\alpha x$, $1-\beta x$ reveal the reciprocal roots α, β. (The roots are $\frac{1}{\alpha}, \frac{1}{\beta}$.)

The roots are the same as the roots of x^2+x-1

i.e. $\frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

The reciprocal roots are $\frac{2}{-1 \pm \sqrt{5}} \cdot \frac{-1 \mp \sqrt{5}}{-1 \mp \sqrt{5}} = \frac{2(-1 \mp \sqrt{5})}{1-5} = \frac{1 \mp \sqrt{5}}{2}$

$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden ratio)

$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$

$\alpha + \beta = 1$

$\alpha - \beta = \sqrt{5}$

$\alpha\beta = -1$

Always use α, β in the algebraic simplification.

$$F(x) = \frac{1+x}{1-x-x^2} = \frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = A\sum_{n=0}^{\infty} (\alpha x)^n + B\sum_{n=0}^{\infty} (\beta x)^n$$

$$= \sum_{n=0}^{\infty} \underbrace{(A\alpha^n + B\beta^n)}_{a_n} x^n$$

$a_n \sim A\alpha^n$ (exponential growth rate)