Combinatorics

Book 2

The graph $\Gamma \setminus v$ (formed by removing v and its edges from Γ) has one fewer vertex, so it can be properly colored using at most 6 colors. And since v has at most 5 neighbors in $\Gamma \setminus v$, there is a color left over which can be used to color vertex v. This gives a proper coloring of Γ using at most 6 colors (a contradiction...)

We will improve this to show that actually 5 colors suffice to properly color every planar graph.

$\psi \ \phi \ \chi \ \pi$

A b c ... $x \ X \ \chi$

Greek "chi"

Given a graph Γ, the chromatic number of Γ, denoted $\chi(\Gamma)$, is the smallest number of colors we can use to properly color the vertices of Γ. A proper coloring of the vertices of Γ is a coloring of the vertices such that no edge has both endpoints of the same color.

The theorem of Appel and Haken (1976) is that every planar graph Γ has $\chi(\Gamma) \leq 4$.

Note that $\chi(K_n) = n$. Here K_n is the complete graph of order n.

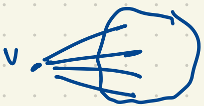
A graph Γ has $\chi(\Gamma) = 1$ iff it has vertices but no edges.

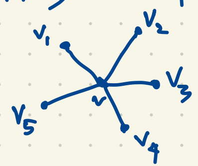
A graph Γ has $\chi(\Gamma) \leq 2$ iff Γ is bipartite iff Γ has no circuits of odd length.

Computing $\chi(\Gamma)$ is hard in general.

Theorem If Γ is a finite planar graph then $\chi(\Gamma) \leq 5$. Proof due to Heawood.

Proof If the theorem fails then there is a smallest counterexample Γ with n vertices (so Γ is planar and every planar graph of order $n-1$ has chromatic number ≤ 5 while $\chi(\Gamma) \geq 6$). We seek a contradiction. Γ has a vertex v of degree ≤ 5. In fact $\deg v = 5$. (If $\deg v \leq 4$ then $\chi(\Gamma) \leq 5$, a contradiction.) Let Γ' be the graph obtained from Γ by deleting v and its five edges,



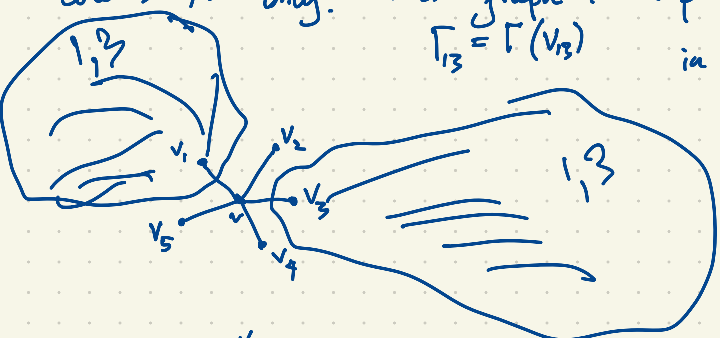


so $\chi(\Gamma') \leq 5$. Say v_i has color i $(i = 1, 2, \cdots, 5)$.

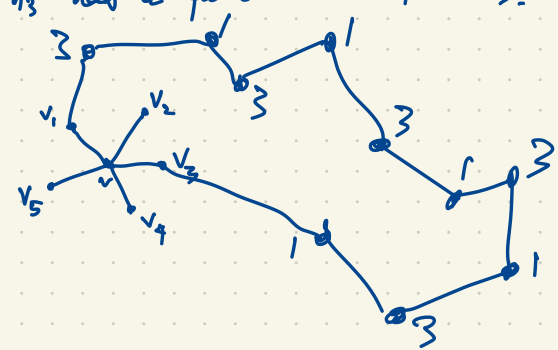
Consider the vertices $V_{13} \subset \{\text{vertices of } \Gamma\}$ having colors 1, 3 only. This graph is bipartite. I can assume v_1 is joined to v_3 in Γ_{13} (otherwise

$\Gamma_{13} = \Gamma(V_{13})$

in part of Γ_{13}, reverse colors 1, 3 so that v_3 gets color 1. Then we are free to color v using color 3 since its neighbors are color 1, 2, 1, 4, 5).

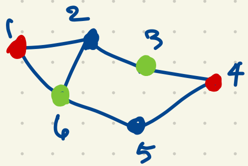


Otherwise Γ_{13} has a path from v_1 to v_3.

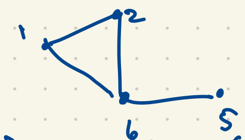


Similarly there is a path from v_2 to v_4 using only vertices of colors 2 and 4. Contradiction! $\square$

Given a graph Γ, a subgraph of Γ is formed by taking a subset of the edges of Γ together with all their vertices. An induced subgraph of Γ is formed by taking a subset of the vertices of Γ together with all their edges in Γ

Γ = 

eg.  is a subgraph of Γ. (not an induced subgraph of Γ)

eg.  is an induced subgraph of Γ.

An induced subgraph of Γ is a subgraph of Γ, but not conversely.

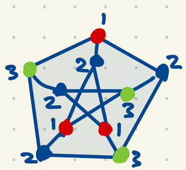
A k-clique in Γ is a complete subgraph of Γ, i.e. a subset of the vertices, any two of which are joined.

In Γ above, $\{1,2,6\}$ is a clique (in fact a 3-clique). The clique number of Γ, denoted $\omega(\Gamma)$, is the size of the largest clique in Γ. It is hard to compute $\omega(\Gamma)$.

w vs. ω
Roman Greek

Theorem For every graph Γ, $\chi(\Gamma) \geq \omega(K)$.

Warning: this not equality! For the Petersen graph P, $\omega(P) = 2$.

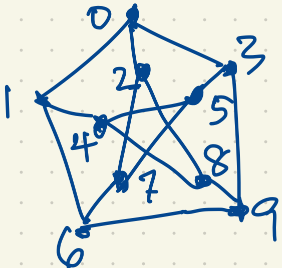


$\chi(P) = 3$ for the Petersen graph P.

Proof: The vertices in a clique of size $\omega(\Gamma)$ require $\omega(\Gamma)$ different colors.

Dual to the clique number $\omega(\Gamma)$ we have the coclique number $\alpha(\Gamma)$ which is the maximum number of vertices in Γ, no two of which are joined. (This is $\alpha(\Gamma) = \omega(\bar{\Gamma})$ where $\bar{\Gamma}$ is the complementary graph). Cocliques are also called independent sets of vertices.

Eg. $\alpha(P) = 4$. $|\chi(P)| \geq \frac{10}{4} = 2.5 \implies \chi(P) \geq 3$.



$\{1,2,3\}$ is a coclique which is not contained in any larger coclique; it is a <u>maximal</u> coclique.

A <u>maximum</u> coclique (i.e. a coclique of maximum size) is $\{1,3,7,8\}$.

This is maximum size because P has vertex set $\{0,3,9,6,1\} \cup \{2,7,5,4,8\}$ as a union of two 5-cycles (circuits of length 5). Any set of size at least 5 vertices has either 3 on the inner 5-cycle $\{2,7,5,4,8\}$ or 3 vertices on the outer 5-cycle $\{0,3,9,6,1\}$. In either case there is an edge in that 5-cycle joining two vertices we have chosen.

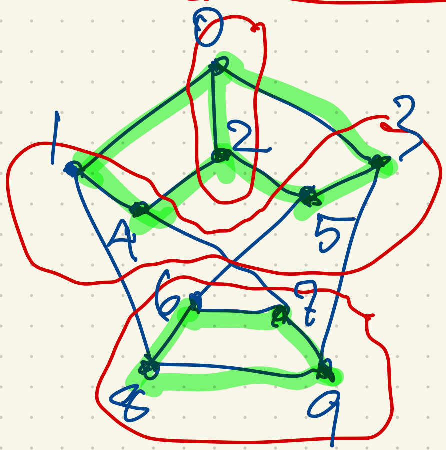
<u>Theorem</u> $\chi(\Gamma) \geq \dfrac{|V|}{\alpha(\Gamma)}$ where $|V|$ = the number of vertices = the order of Γ.

<u>Proof</u> Let $k = \chi(\Gamma)$. Properly color the vertices $1, 2, \cdots, k$ and let V_i be the subset of vertices colored i, for $i = 1, 2, \cdots, k$. This gives a partition $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ ($V_i \cup V_j$ = union of V_i and V_j; $V_i \sqcup V_j$ = disjoint union of V_i and V_j). Each V_i is a coclique so $|V_i| \leq \alpha(\Gamma)$ so $|V| = |V_1| + |V_2| + \cdots + |V_k| \leq \underbrace{\alpha(\Gamma) + \alpha(\Gamma) + \cdots + \alpha(\Gamma)}_{k} = k\,\alpha(\Gamma)$ so $\chi(\Gamma) = k \geq \dfrac{|V|}{\alpha(\Gamma)}$. $\square$

March	M	W	F
		1	3
	6	$\boxed{8}$	10
	13	15	17

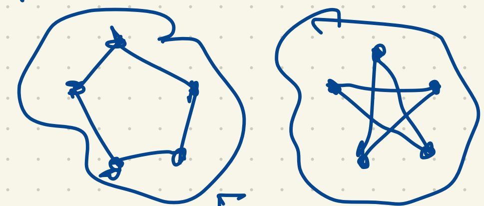
Spring Break

Test 1: Wed Mar 8

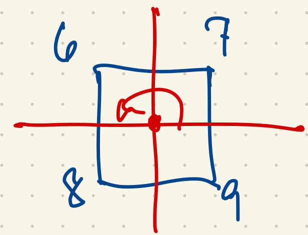


$= \Gamma$

You can use nauty to test isomorphism between two graphs.



Γ $\quad$ Γ'



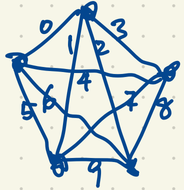
Using nauty software

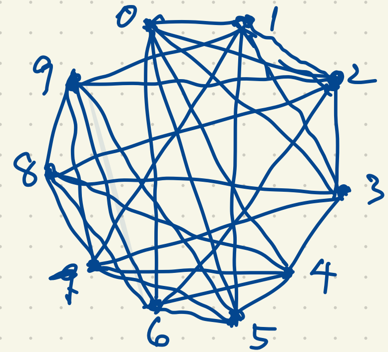
$G = \operatorname{Aut} \Gamma$

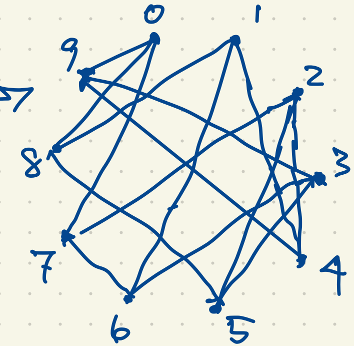
$G = \langle (1,3)(4,5)(6,7)(8,9),\ (0,2)(1,4)(3,5)(6,9)(7,8) \rangle$

$|G| = 4$

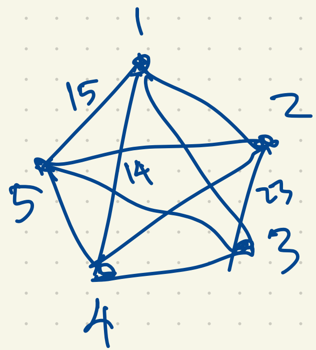
G has 3 orbits on the vertices: $\{0,2\}, \{1,3,4,5\}, \{6,7,8,9\}$

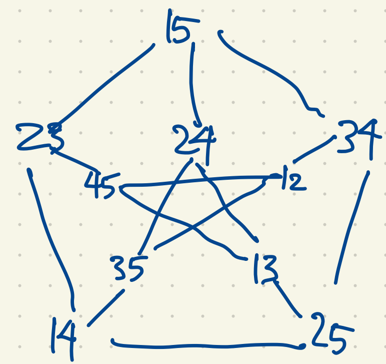
$K_5 =$ 

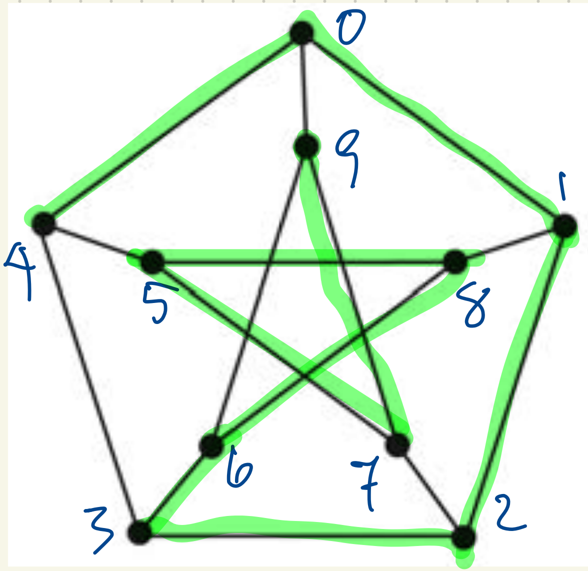
$\longrightarrow$ K_5' 

$\xrightarrow{\overline{K_5'}}$ 

$|\mathrm{Aut}\ \overline{K_5'}| = 120$

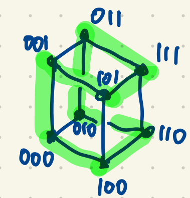
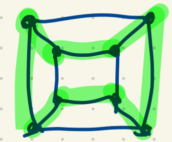
$K_5 =$ 

$\overline{K_5'} =$ 



The Petersen graph P has a Hamilton path

$(0,1,2,3,6,8,5,7,9)$ (a path touching each vertex exactly once) but no Hamilton circuit (ending at the same vertex where it started).

The Hamming cube H_3 =  = 

does have a Hamilton circuit.

000
100
110
010
011
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001
000

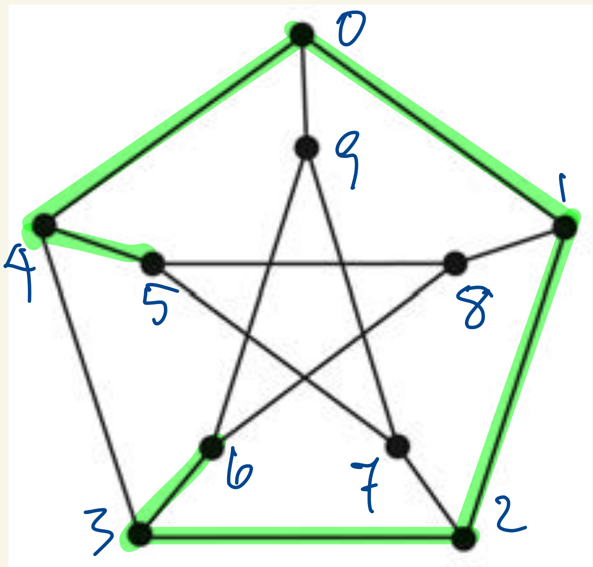
"Gray code"

Every Hamming graph H_n $(n \geq 2)$ has a Hamilton circuit.

A graph having a Hamilton circuit is called <u>Hamiltonian</u>.

Looking for Hamilton paths or circuits is known to be difficult in general.

Testing whether a giving graph Γ is Hamiltonian is NP-complete.

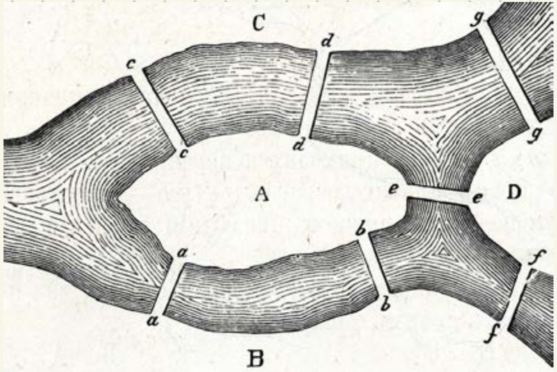


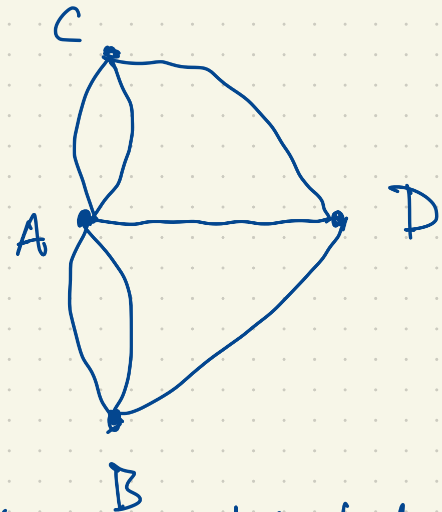
<u>Theorem</u>: The Petersen graph P is not Hamiltonian, i.e. it does not have a Hamilton circuit/cycle.

<u>Proof</u> Suppose P has a Hamilton circuit. Without loss of generality this circuit contains the path (4, 0, 1, 2). (This is because P has 120 automorphisms mapping any such path of length 3 to any other.) The Hamilton circuit uses two of the edges from vertex 3, so it uses either {3,4} or {2,3}; so without loss of generality, it uses the edge {2,3}. We cannot use the edge {3,4} as this would complete the circuit without passing through all vertices; So we must use the edges {3,6} and {4,5}. To continue the circuit from vertex 6, we have two choices: proceed through vertex 8 or vertex 9. Neither of these choices leads to a Hamilton circuit. This is a contradiction. $\square$

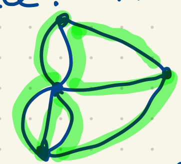
Euler paths and circuits

The Seven bridges of Königsberg





An <u>Euler</u> <u>trail</u> is a trail (repeating vertices but not edges) which uses each edge exactly once. An <u>Euler circuit</u> is an Euler trail that returns to its starting point.



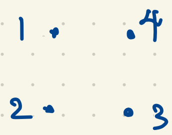
This graph has an Euler trail. In order to have an Euler trail, a graph must have either 0 or 2 vertices of odd degree. When there are no vertices of odd degree, we have an Euler circuit.

<u>Theorem</u> (Euler) A graph has an Euler trail iff it is connected and it has either 0 or 2 vertices of odd degree. In the case every vertex has even degree, we have an Euler circuit/cycle.

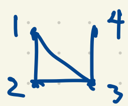
We sometimes speak of labelled graphs and unlabelled graphs.

Eg. on the vertex set $\{1,2,3,4\}$, there are $2^6 = 64$ labelled graphs

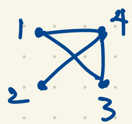
$\binom{4}{2} = 6$ pairs of vertices.



There are $\binom{n}{2}$ labelled graphs on n vertices.
But many of them are isomorphic.

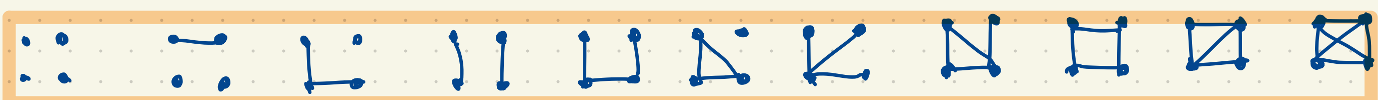


$\neq$



These are different labelled graphs but they are isomorphic.

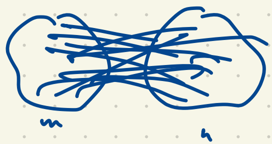
As unlabelled graphs they are isomorphic, hence the same graph.



There are 11 unlabelled graphs of order 4 i.e. 11 isomorphism types of graphs of order 4, i.e. 11 graphs of order 4 up to isomorphism.

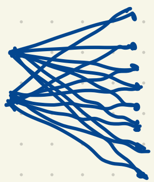
The Petersen graph has girth 5 (the shortest cycle has length 5). It has 15 edges.
For a graph on 10 vertices, 15 edges is the maximum possible for girth 5.
For a graph on 10 vertices without triangles (i.e. girth ≥ 4), what is the maximum possible number of edges ?

Recall: $K_{m,n}$ =



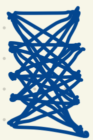
$K_{m,n}$ has mn edges. $K_{m,n}$ is bipartite so it has no cycles of odd length. In particular it has no triangles.

$K_{2,8}$ has 16 edges



$K_{2,8}$

16 edges



$K_{5,5}$

25 edges
girth 4
(no triangles)

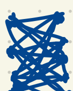
<u>Theorem</u> (Mantel 1907)
If Γ is a graph of order n with no triangles (i.e. its girth is at least 4) then Γ has at most $\frac{n^2}{4}$ edges.

If n is even then $K_{\frac{n}{2},\frac{n}{2}}$ attains the upper bound of $\frac{n^2}{4}$ edges. What if n is odd?

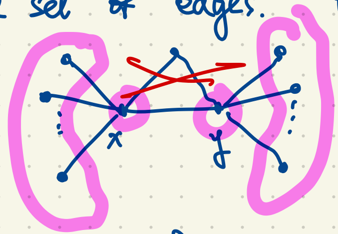
$$\left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even, for } K_{\frac{n}{2},\frac{n}{2}} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd, for } K_{\frac{n+1}{2},\frac{n-1}{2}} \end{cases}$$

On 9 vertices, any graph without triangles has at most <u>20</u> edges.

$K_{4,5}$



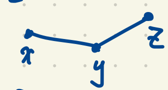
Proof Let Γ be a graph of order n with no triangles, $\Gamma = (V, E)$. (V is the set of vertices, E is the set of edges.) For every edge $\{x,y\} \in E$, $d(x) + d(y) \leq n$.



$$d(x) - \cancel{1} + \cancel{1} + \cancel{1} + d(y) - \cancel{1} \leq n$$

Add the inequality $d(x)+d(y) \leq n$ over all edges $\{x,y\} \in E$ to get $\sum_{\{x,y\}\in E} (d(x)+d(y)) \leq ne$.

Next, count the number of triples of vertices (x,y,z) with $x \sim y \sim z$.

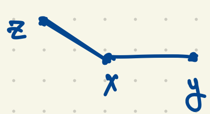


There are n choices for $y \in V$ and $d(y)$ choices for x, $d(y)$ choices for z, so $d(y)^2$ choices for x and z (given y). The total number of walks of length 2 is $\sum_{y \in V} d(y)^2$.

On the other hand, there are $e = |E|$ edges in Γ. For the edge $\{x,y\} \in E$, how many walks of length 2 contain this edge? $d(x) + d(y)$ choices of walk of length 2 in which we include a step from x to y

The total number of walks of length 2 is

$$\sum_{\{x,y\}\in E} (d(x)+d(y)).$$



$d(x)$ choices for z (given the edge $\{x,y\}$)

or

$d(y)$ choices for z (given the edge $\{x,y\}$)

Therefore

$$\sum_{x\in V} d(x)^2 = \sum_{\{x,y\}\in E} \underbrace{(d(x)+d(y))}_{\leq n} \leq en$$

If $\sum_{x\in V} d(x) = 2e$, what does this tell us about $\sum_{x\in V} d(x)^2$?

Use the Cauchy-Schwarz inequality.

$$(\underbrace{d(1)+\cdots+d(n)}_{2e})^2 \leq n(d(1)^2+\cdots+d(n)^2)$$

$$4e^2 \leq n\sum_{x\in V} d(x)^2 = n\sum_{\{x,y\}\in E}(d(x)+d(y)) \leq n\cdot ne = n^2e$$

So $e \leq \frac{n^2}{4}$.

Cauchy-Schwarz Inequality

Given two vectors

$a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$

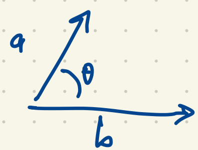
$b = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$,

$$|a\cdot b| \leq \|a\|\cdot\|b\|$$

where $a\cdot b = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$\|a\| = \sqrt{a_1^2+a_2^2+\cdots+a_n^2}$

$$a\cdot b = \|a\|\|b\|\cos\theta$$



$|\cos\theta| \leq 1$.

Special case: $b = (1,1,1,\cdots,1)$

$\|b\| = \sqrt{\underbrace{1^2+1^2+\cdots+1^2}_{n}} = \sqrt{n}$

$a\cdot b = a_1+a_2+\cdots+a_n$

$|a\cdot b| \leq \|a\|\cdot\sqrt{n}$

$$(a_1+\cdots+a_n)^2 \leq n(a_1^2+\cdots+a_n^2)$$

Proof of Cauchy-Schwarz: Fix $a, b \in \mathbb{R}^n$. Consider

$$f(t) = \|a - tb\|^2 = (a-tb)\cdot(a-tb) = a\cdot a - tb\cdot a - ta\cdot b + t^2\, b\cdot b = \|a\|^2 - 2t(a\cdot b) + t^2\|b\|^2.$$

Of course $f(t) \geq 0$ for all t.

disc < 0

or

disc $= 0$

NOT

disc > 0

The discriminant

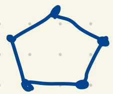
$$(-2a\cdot b)^2 - 4\|a\|^2\|b\|^2 \leq 0$$

$$4(a\cdot b)^2 \leq 4\|a\|^2\|b\|^2$$

$$(a\cdot b)^2 \leq \|a\|^2\|b\|^2.$$

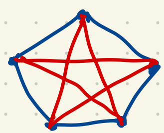
The disc. of at^2+bt+c is b^2-4ac

Show me a graph Γ of order 5 such that neither Γ nor $\bar{\Gamma}$ has a triangle.
(i.e. both Γ and $\bar{\Gamma}$ have girth ≥ 4). Recall: $\bar{\Gamma}$ is the complement of Γ.

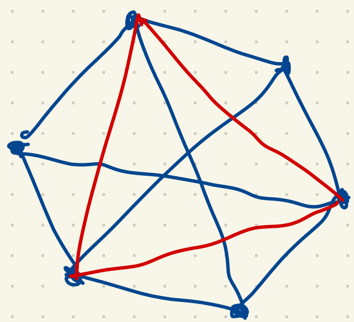
$\Gamma =$  $\bar{\Gamma} =$  $\cong \Gamma$ (a 5-cycle). Both Γ and $\bar{\Gamma}$ have girth 5.

Show me a graph Γ of order 6 such that neither Γ nor $\bar{\Gamma}$ have a triangle.
There is no such graph. Why not?

Color the edges of K_5 with 2 colors red, blue. To avoid a monochromatic triangle (all red or all blue):

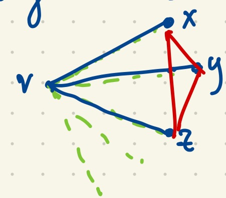


K_6 :



<u>Theorem</u> If we color the edges of K_6 red and blue, then there is either a red triangle or a blue triangle.

<u>Proof</u> Consider a vertex v.



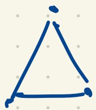
There are five edges from v so by the Pigeon hole Principle, at least three of them are the same color, say $\{v,x\}$, $\{v,y\}$, $\{v,z\}$ are blue.

Now x, y, z form a triangle.
If any edge of this triangle is blue then together with the edges to v we have a blue triangle. Otherwise all edges of the triangle $\{x,y,z\}$ are red. $\square$

Theorem (Ramsey) Let r, s be positive integers. There is a number $R(r,s)$ such that for all $n \geq R(r,s)$, every 2-coloring (red, blue) of the edges of K_n has either a red r-clique or a blue s-clique. For $n < R(r,s)$, there exists a coloring of the edges of K_n with 2 colors (red, blue) having no red r-clique and no blue s-clique.

$R(3,3) = 6.$

$R(2,3) = 3$



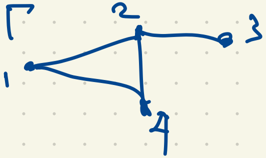
$43 \leq R(5,5) \leq 48$

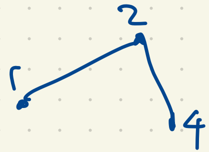
$R(r,s)$ values are Ramsey numbers.

(Extremal graph theory)

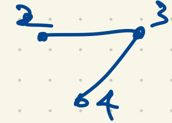
Values / known bounding ranges for Ramsey numbers $R(r, s)$ (sequence A212954 in the OEIS)

r \ s	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–42
4				**18**	25[9]	36–40	49–58	59[13]–79	73–106	92–136
5					**43–48**	58–85	80–133	101–194	133–282	149[13]–381
6						**102–161**	115[13]–273	134[13]–427	183–656	204–949
7							**205–497**	219–840	252–1379	292–2134
8								**282–1532**	329–2683	343–4432
9									**565–6588**	581–12677
10										**798–23556**

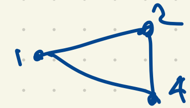




is a subgraph of Γ



is not a subgraph.

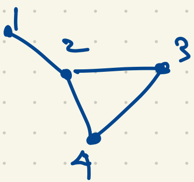


is an induced subgraph.

Graph Theory $\longleftrightarrow$ Linear Algebra

Matroid Theory

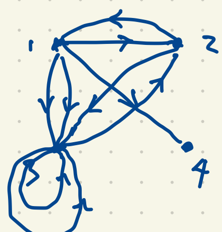
The adjacency matrix of a graph Γ with vertices $1, 2, 3, \ldots, n$ is the $n \times n$ matrix whose (i,j)-entry is the number of edges from vertex i to vertex j. Eg.

$\Gamma =$  has adjacency matrix

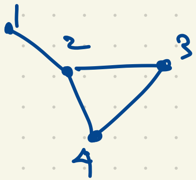
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = A$$

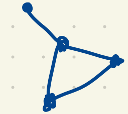
Symmetric $(0,1)$-matrix with "zero diagonal" (corresponding to an undirected graph with no loops or multiple edges)

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



directed graph with loops and multiple edges

$\Gamma =$  has adjacency matrix $\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 \end{array} = A$

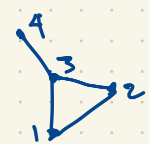
The unlabelled graph  has several choices of adjacency matrix.

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 1 & 1 \\ 2 & 0 & 3 & 1 & 1 \\ 3 & 1 & 1 & 2 & 1 \\ 4 & 1 & 1 & 1 & 2 \end{array}$$

The (i,j)-entry of A^2 is the number of walks of length 2 from vertex i to vertex j.
(A walk is allowed to repeat edges or vertices, unlike in a path or a trail.)

$$A^3 = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}}_{A^2}\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 0 & 3 & 1 & 1 \\ 3 & 2 & 4 & 4 \\ 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 2 \end{bmatrix}$$

The (i,j)-entry of A^m is the number of walks of length m from vertex i to vertex j in Γ.

 has adjacency matrix $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$A^0 = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A^1 = A$$

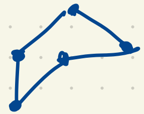
$$J = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad j = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

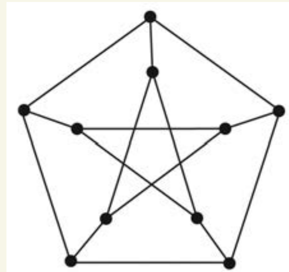
If Γ is a k-regular graph then $Aj = kj$, $AJ = kJ$
i.e. j is an eigenvector of A with eigenvalue k.

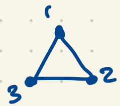
$$Aj = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$ degree sequence

$$AJ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

Which graphs are connected regular with diameter 2 and girth 5? Any such graph has order 5, 10, 50 or 3250.

5-cycle 

Petersen graph 

Look at $K_3 =$  with adjacency matrix $A = \begin{array}{c|ccc} i \backslash j & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{array}$. The number of walks of length k from vertex i to vertex j is the (i,j)-entry of A^k.

How many walks of length 10 are there from vertex 2 to vertex 3 in K_3? 341.

$$A^{10} = \begin{bmatrix} 342 & 341 & 341 \\ 341 & 342 & 341 \\ 341 & 341 & 342 \end{bmatrix}$$

Since A is a real symmetric matrix, it is diagonalizable i.e. there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A. One obvious eigenvector $\underline{u}_1 = \underline{j} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: $A\underline{j} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\underline{j}$ since K_3 is 2-regular. Another eigenvector $\underline{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$: $A\underline{u}_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -\underline{u}_2$

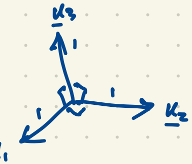
$\underline{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $A\underline{u}_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = -\underline{u}_3$

It is customary to normalize the eigenvectors as

$$\underline{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \underline{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \underline{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

so that $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

Orthonormal means $\underline{u}_i \bullet \underline{u}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$



$A\underline{u}_i = \lambda_i \underline{u}_i \qquad \lambda_1 = 2, \quad \lambda_2 = \lambda_3 = -1.$

A has diagonal form $D = \begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$. $\quad D^{10} = \begin{bmatrix}1024&0&0\\0&1&0\\0&0&1\end{bmatrix}$

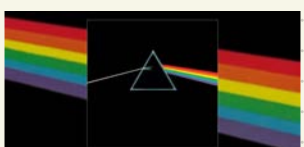
$A = UDU^{-1}$ i.e. $U^{-1}AU = D$ where $U = \begin{bmatrix}\underline{u}_1 & \underline{u}_2 & \underline{u}_3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt3} & \frac{1}{\sqrt2} & \frac{1}{\sqrt6}\\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt6}\\ \frac{1}{\sqrt3} & 0 & -\frac{2}{\sqrt6}\end{bmatrix}$

$A^{10} = (UD\cancel{U^{-1}})(\cancel{U}D\cancel{U^{-1}})(\cancel{U}D\cancel{U^{-1}})\cdots(\cancel{U}DU^{-1}) = UD^{10}U^{-1} = \begin{bmatrix}342&341&341\\341&342&341\\341&341&342\end{bmatrix}$

$(x-2)(x^2+2x+1)$
$= x^3 - 0x^2 - 3x - 2$

$A = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix} \xrightarrow{\text{diagonalize}} D = \begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$

Check: $\det A = 2 \qquad \operatorname{tr} A = 0$
$\det D = 2 \qquad \operatorname{tr} D = 0$



The list of eigenvalues 2, −1, −1 is the spectrum of A (or of D)

A, D are similar so they have the same trace, determinant and characteristic polynomial

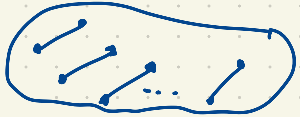
$\det(xI - A) = \det(xI - D) = (x-2)(x+1)^2$
$= x^3 - 0x^2 - 3x - 2$

$\begin{bmatrix}1&1\\0&1\end{bmatrix}$ has eigenvect[or] $\begin{bmatrix}1\\0\end{bmatrix}$ but no basis of eigenvectors.

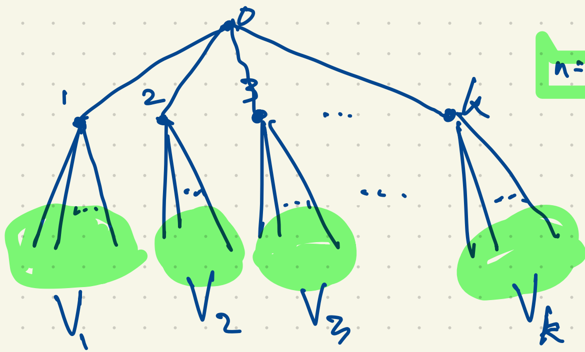
$\begin{bmatrix}3&1\\0&2\end{bmatrix}$ has a basis of eigenvectors $\begin{bmatrix}1\\0\end{bmatrix}$ (with eigenvalue 3) $\begin{bmatrix}1\\-1\end{bmatrix}$ (with eigenvalue 2) but they aren't perpendicular/orthogonal.

A <u>Moore graph</u> is a k-regular graph of diameter 2 and girth 5. What are the examples?

$k \geq 2$



1-regular graphs are not connected unless $n=2$.

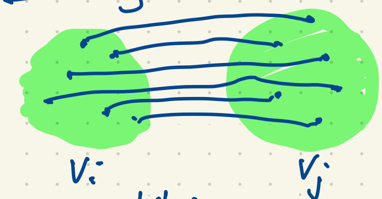


$$\boxed{n = k+1 + k(k-1) = k^2+1}$$

$|V_1| = k-1$

$$|V_i| = k-1$$

Between V_i and V_j $(i \neq j)$ the edges form a <u>perfect matching</u>



$\left((k-1)!\right)^{\frac{k(k-1)}{2}}$ ways to choose these perfect matchings.

If you can do this without creating any 3-cycles or 4-cycles then you have a Moore graph

Now let A be the adjacency matrix of the Moore graph of degree k, $n = k^2+1$. A is $n \times n$, $I = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n}$

$$J = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n} \qquad \underline{j} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$$

$$A\underline{j} = k\underline{j} \qquad AJ = kJ$$
$$JA = kJ$$

$A^T = A$

$(AJ)^T = (kJ)^T$

$JA = J^TA^T = kJ$

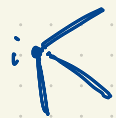
A, J, I all commute with each other.

$J-I-A$ is the adjacency matrix of the complementary graph.

$\boxed{A^2 = kI + (J - I - A)}$

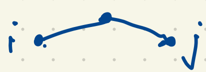
why? The (i,j)-entry of A^2 is the number of walks of length 2 from vertex i to vertex j.

Case $i=j$: A^2 has k in the diagonal entry (i,i).



Case $i \sim j$ 

. Here the (i,j)-entry of A^2 is 0. Same on the right hand side.

Case $i \not\sim j$, $i \neq j$ 

Here the (i,j)-entry on both sides is 1.

Since A is a real symmetric $n \times n$ matrix, it has n eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ with corresponding eigenvectors $\underline{u}_1, \cdots, \underline{u}_n$ i.e. $A\underline{u}_i = \lambda_i \underline{u}_i$. Some of the eigenvalues will be repeated. One of the eigenvalues is k since $A\underline{j} = k\underline{j}$, with eigenvector $\underline{j} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ or $\frac{1}{\sqrt{n}}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. What are the other eigenvalues and eigenvectors?

Say $\underline{u}$ is an eigenvector different from $\frac{1}{\sqrt{n}}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, with corresponding eigenvalue λ so $A\underline{u} = \lambda \underline{u}$.

So $\underline{u}$ is perpendicular (orthogonal) to $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ i.e. $J\underline{u} = \underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$A^2 \underline{u} = A A \underline{u} = A \lambda \underline{u} = \lambda A \underline{u} = \lambda^2 \underline{u}$$

$$A^2 \underline{u} = [kI + (J - I - A)]\underline{u}$$

$$\lambda^2 \underline{u} = k\underline{u} - \underline{u} - \lambda \underline{u}$$

$$\lambda^2 = k - 1 - \lambda$$

$$\lambda = k - 1 - \lambda^2 < k$$

So $\lambda_1 = k$ and $\lambda_2, \cdots, \lambda_n$ are less than k.

Since we have $n-1$ remaining eigenvalues $\lambda_2, \dots, \lambda_n$ all of which are roots of $\lambda^2+\lambda-(k-1)=0$, some of the eigenvalues must be repeated. They are

$$\tfrac{1}{2}\left(-1 \pm \sqrt{1+4(k-1)}\right) = \tfrac{1}{2}\left(-1 \pm \sqrt{4k-3}\right)$$

$$\lambda = \tfrac{1}{2}\left(-1+\sqrt{4k-3}\right)$$

$$\mu = \tfrac{1}{2}\left(-1-\sqrt{4k-3}\right)$$

So the spectrum of A is $k, \underbrace{\lambda, \lambda, \dots, \lambda}_{m}, \underbrace{\mu, \mu, \dots, \mu}_{n-1-m}$. Since $\operatorname{tr} A = 0$,

$$k + m\lambda + (n-1-m)\mu = 0.$$

This allows us to solve for m.

$n = k^2+1$

$$2k + m\left(-1+\sqrt{4k-3}\right) + (n-1-m)\cdot\left(-1-\sqrt{4k-3}\right) = 0$$

$$2k + -\underbrace{(n-1)}_{k^2} + \underbrace{(2m-n+1)}_{2m-k^2}\sqrt{4k-3} = 0$$

$$(2m-k^2)\sqrt{4k-3} = k(k-2). \quad (*)$$

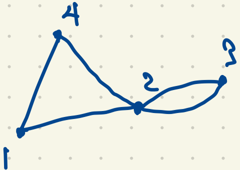
If $k=2$ we get $\lambda = \tfrac{1}{2}(-1+\sqrt{5})$, $\mu = \tfrac{1}{2}(-1-\sqrt{5})$ and the spectrum is $2, \underbrace{\lambda, \lambda}_{m=2}, \mu, \mu$

Otherwise $k>2$ so $4k-3$ must be an integer square i.e. $4k-3 = d^2$ where d is an integer, $d>0$. $k = \dfrac{d^2+3}{4}$. Substitute into $(*)$ to get

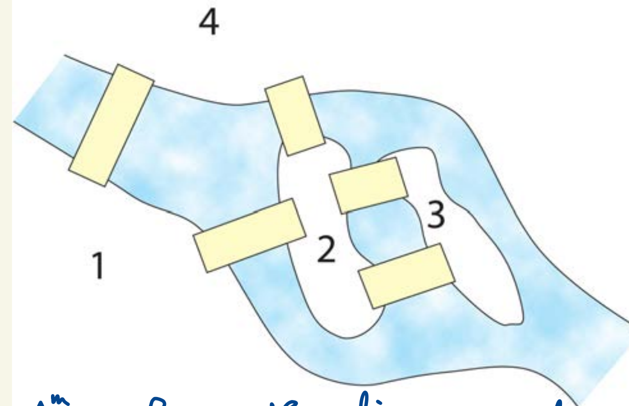
$$(d^4+d^3+6d^2-2d+9-32m)d = 15. \Rightarrow d = 1, 3, 5 \text{ or } 15. \Rightarrow k = \cancel{1}, 3, 7 \text{ or } 57.$$

Counting walks in graphs

How many ways can we walk from land mass i to land mass j by crossing m bridges?



Γ



This number $w_m(i,j)$, equals the number of walks of length m in Γ. It is also the (i,j)-entry of A^m where the adjacency matrix of Γ is

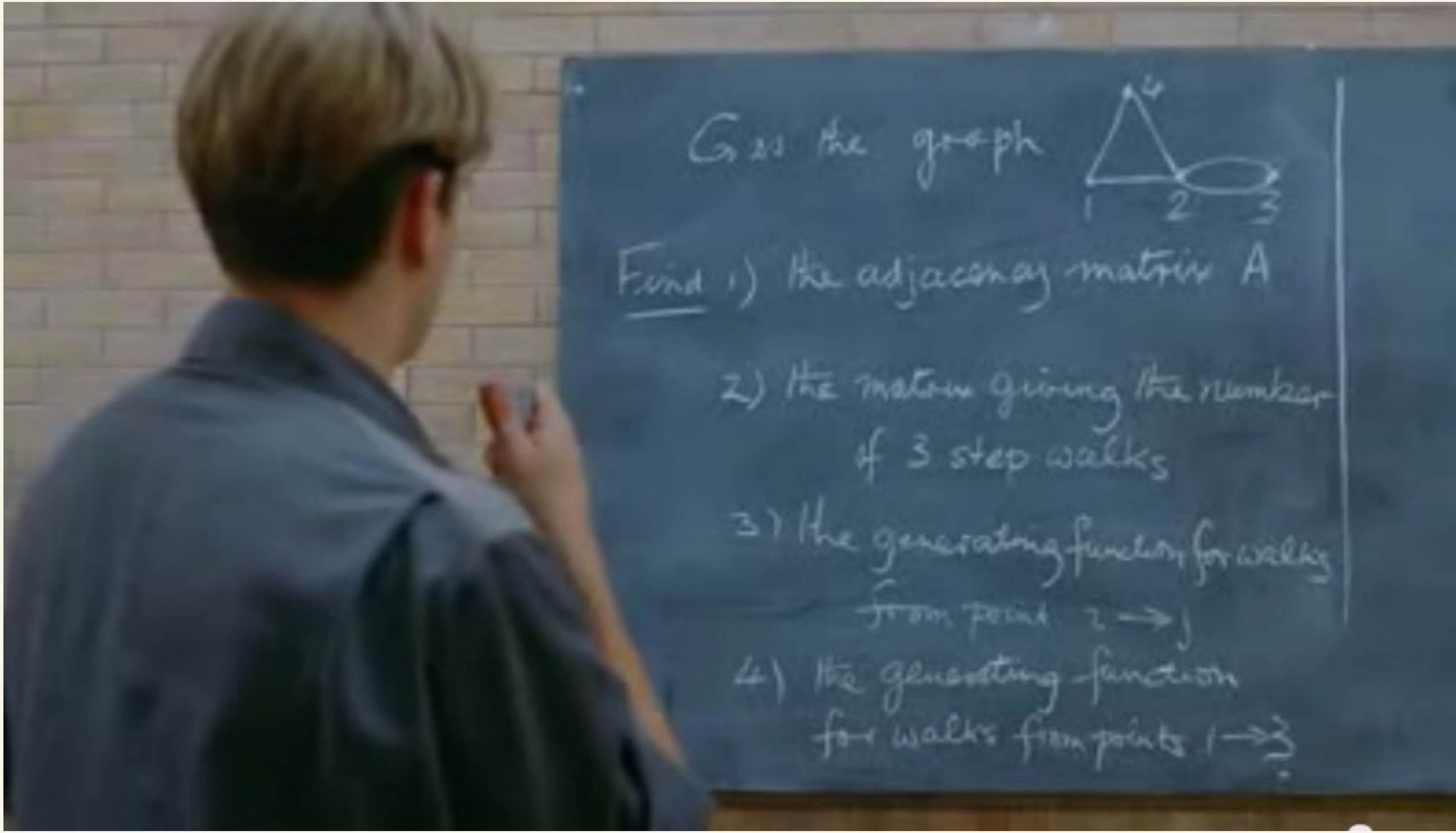
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Rather than working out A^m in each case, we want a "formula" for $w_m(i,j)$.
We are able to give a closed formula for

$$W_{ij}(x) = \sum_{m=0}^{\infty} w_m(i,j)x^m = w_0(i,j) + w_1(i,j)x + w_2(i,j)x^2 + w_3(i,j)x^3 + \cdots$$

which is the <u>generating function</u> for the sequence $w_0(i,j), w_1(i,j), w_2(i,j), \ldots$
$W_{ij}(x)$ is the (i,j)-entry of the matrix

$$W(x) = \sum_{m=0}^{\infty} A^m x^m = I + Ax + A^2x^2 + A^3x^3 + A^4x^4 + \cdots$$


G is the graph
4
1 2 3
Find 1) the adjacency matrix A
2) the matrix giving the number
of 3 step walks
3) the generating function for walks
from point i → j
4) the generating function
for walks from points 1 → 3

Recall: A geometric series $1+r+r^2+r^3+\cdots = \frac{1}{1-r}$ if $|r|<1$.

But in a formal power series, convergence is a non-issue because the terms are purely symbolic, not numbers.

Check: cross-multiply. $(1-r)(1+r+r^2+r^3+\cdots) = 1-\cancel{r}+\cancel{r}-\cancel{r^2}+\cancel{r^2}-\cancel{r^3}+\cancel{r^3}-\cancel{r^4}+\cancel{r^4}-\cdots = 1.$

$$W(x) = I + Ax + A^2x^2 + A^3x^3 + \cdots = (I-Ax)^{-1}$$

Check: $(I-Ax)(I+Ax+A^2x^2+A^3x^3+\cdots) = I - \cancel{Ax} + \cancel{Ax} - \cancel{A^2x^2} + \cancel{A^2x^2} - \cancel{A^3x^3} + \cancel{A^3x^3} - \cdots$

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$I - Ax = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & -x & 0 & -x \\ -x & 1 & -2x & -x \\ 0 & -2x & 1 & 0 \\ -x & -x & 0 & 1 \end{bmatrix}$$

$(I-Ax)^{-1} = \frac{1}{d(x)} \operatorname{adj}(I-Ax)$ where $d(x) = \det(I-Ax)$ and $\operatorname{adj}(I-Ax)$ is the classical adjoint of $I-Ax$.

We won't work out 4×4 inverses of matrices with polynomial entries by hand.

$$(I-Ax)^{-1} = W(x) = \frac{1}{d(x)} \begin{bmatrix} 1-5x^2 & x(1+x) & 2x^2(1+x) & \cdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where $d(x) = 4x^4 + \cdots$

The $(1,3)$-entry is $W_{13}(x) = \dfrac{2x^2(1+x)}{d(x)} = \dfrac{2x^2}{1-x-6x^2+4x^3} = 2x^2 + 2x^3 + 14x^4 + 18x^5 + 94x^6 + 146x^7 + 638x^8 + \cdots$